Mathematical Writing Exercises

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For each of the questions below, work on the problem, then read the given solutions, and decide whether or not the solutions given are well written. In the cases that they are not, decide how you would improve them. It may help to discuss with your peers whether they are easy or difficult to read, and whether or not you understand what the writer is attempting to explain. Compare these solutions to your own solutions, and see if there is anything in your own solutions you would change in light of what made these ones difficult to understand.

1 Question 1

Let n be an odd integer with $n > 3$. Let $M = n^2 + 2n - 7$. Prove that M always has at least four different integer factors (excluding 1 and M).

1.1 Solution 1

$M = (n+1)^2 - 8$ and n is odd so M is a multiple of 4, so the factors are $2, 4, \frac{M}{4}, \frac{M}{2}$ (since even number^2 =even and a multiple of 4 because the factor of 2 is counted twice in the square). The factors are all different because $M \geq 28$ by definition of n.

1.2 Solution 2

Let $n = 2k + 1$ for an integer k. We can do this because n is odd, and therefore is one more than an even number.

Then,

$$\begin{aligned}
M &= (2k+1)^2 + 2(2k+1) - 7 \\
&= 4k^2 + 4k + 1 + 4k + 2 - 7 \\
&= 4k^2 + 4k - 4 \\
&= 4(k^2 + k - 1).
\end{aligned}$$

So, M has factors 4 and $k^2 + k - 1$, and also 2 and $2(k^2 + k - 1)$. Since $n \geq 5$, $k \geq 2$ so $k^2 + k + 1 \geq 7$ and therefore $M \geq 28$. Moreover, all of these factors are at most $\frac{M}{2}$. So they are all different from each other and different from 1 and M.

1.3 Solution 3

Rewriting $M = n^2 + 2n - 7 = n^2 + 2n + 1 - 8 = (n+1)^2 - 8$. If n is an odd number, then $n+1$ is even. Therefore, 2 is a factor of $(n+1)^2$. Moreover, $(n+1)^2$ is what we get if we square all of the factors of $(n+1)$ and multiply them together, so in fact $(n+1)^2$ is divisible by 4. So, since 8 is also divisible by 4, M must be divisible by 4. So, $2, 4, \frac{M}{4}, \frac{M}{2}$ are all factors of M. Since $n \geq 5$, $M \geq 6^2 - 8 = 28$ so all of these factors are different from each other and different from 1 and M.

1.4 Solution 4

$M = 4(k^2 + k + 1)$ so it is divisible by 4 so we just need to check that $2, 4, \frac{M}{2}, \frac{M}{4}$ are different. This is because $M \geq 28$ so $\frac{M}{4} \geq 7 > 2$.

1.5 Solution 5

Rewriting, $M = n^2 + 2n - 7 = n^2 + 2n + 1 - 8 = (n+1)^2 - 8$. Since n is odd, $n+1$ is even, so $n+1 = 2m$ for some whole number m. Therefore, $(n+1)^2 = 4m^2$ so $(n+1)^2$ is a multiple of 4.

Since $M = (n+1)^2 - 8$, and both $(n+1)^2$ and 8 are multiples of 4, M must also be a multiple of 4, so $2, 4, \frac{M}{2}$ and $\frac{M}{4}$ are all factors of M.

Since $n > 3$ and n is odd, $n \geq 5$ so using our expression for M on the first line, $M \geq 6^2 - 8$ so $M \geq 28$. Therefore, $\frac{M}{4} \geq 7$, and in particular $\frac{M}{4} > 4$ so $1 < 2 < 4 < \frac{M}{4} < \frac{M}{2} < M$, so $2, 4, \frac{M}{4}$ and $\frac{M}{2}$ are all different factors of M and none of them are equal to 1 or M.

2 Question 2

Let n be an integer, with $n \geq 2$. Suppose you have statements $P_1, P_2, \cdots, P_n$, and you wish to show that all of these statements are equivalent. What is the smallest number of statements of the form $P_i \implies P_j$ that you must prove in order to show the full equivalence?

2.1 Solution 1

At least n of them are needed because if we don't have n then there must be some P_i that doesn't imply any of the others so they can't all be equivalent, and n is enough because

$$P_1 \implies P_2, \quad P_2 \implies P_3, \quad \ldots, \quad P_{n-1} \implies P_n, \quad P_n \implies P_1$$

works.

2.2 Solution 2

We claim first that n statements is enough. To see this,

$$P_1 \implies P_2, \quad P_2 \implies P_3, \quad \ldots, \quad P_{n-1} \implies P_n, \quad P_n \implies P_1 \quad (\star)$$

works because if we want to deduce $P_i \implies P_j$ then we can chain together

$$P_i \implies P_{i+1} \implies \ldots \implies P_{j-1} \implies P_j$$

(where we regard $n+1$ as the same thing as 1), so n statements is enough.

Now we claim that n statements are necessary. To see this, if we delete any one of our n statements in $(\star)$ then we can't deduce that they are all equivalent any more: by symmetry it doesn't matter which of the n we delete, so without loss of generality suppose we delete $P_1 \implies P_2$. Then, we can't deduce that P_1 and P_2 are equivalent because P_1 does not appear on the left of any of our statements, and P_2 does not appear on the right of any of them, so any chain of implications we can write down will not start with P_1 or end with P_2.

2.3 Solution 3

Need everything to appear on the left so at least n. So the answer is n.

2.4 Solution 4

We claim first that n statements is enough. To see this,

$$P_1 \implies P_2, \quad P_2 \implies P_3, \quad \ldots, \quad P_{n-1} \implies P_n, \quad P_n \implies P_1 \quad (\star)$$

works because if we want to deduce $P_i \implies P_j$ then we can chain together

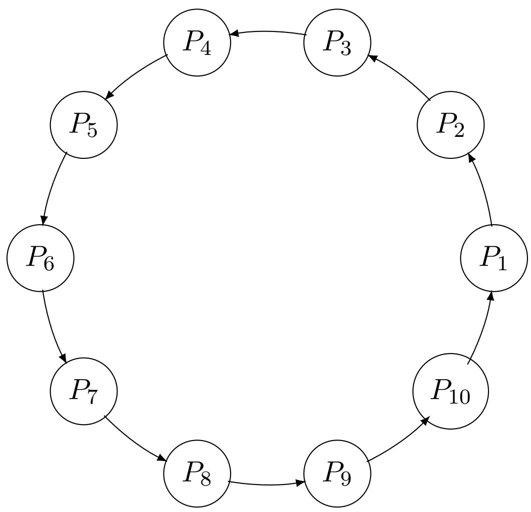
$$P_i \implies P_{i+1} \implies \ldots \implies P_{j-1} \implies P_j$$

(where we regard $n+1$ as the same thing as 1), so n statements is enough.

Now we claim that n statements are necessary. Suppose that we have proved less than n statements of the form $P_i \implies P_j$. Then there must be some i between 1 and n such that we have not proved the statement $P_i \implies P_j$ for all j. This is because, if we had proved a statement with P_i on the left for all i, we would have proved one with P_1 on the left, one with P_2 on the left, etc, up to P_n, and that would be at least n statements. Now, if we can deduce that P_i is equivalent to P_{i+1} then we must be able to chain together some implications with P_i on the extreme left and P_{i+1} on the extreme right, but this isn't possible because we don't have a statement with P_i on the left to start off with.

2.5 Solution 5

If we have a cycle of n implications proved, we can always travel around the cycle from P_i until we hit P_j, e.g. when $n = 10$ this looks like



so n is always enough. The only thing that could stop a cycle is if we can't get from some P_i to some P_j, as otherwise we can build a cycle. So if we have enough statements to show they are all equivalent we must have a cycle and the shortest cycle is of length n, so the answer is n.