Lie Algebras and their Representations

Notes by Eve Pound*

Based on lectures by Dr Beth Romano

1 Lie Algebras in the Wild

1.1 Where do Lie algebras come from?

A Lie group is (essentially) a group that is also a smooth manifold, for example $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{SO}_n, \mathrm{Sp}_{2n}$. Let G be a Lie group. Then the Lie algebra of G is the tangent space at the identity $\mathfrak{g} = T_e G$. $\mathfrak{g}$ is a vector space with additional structure.

By taking a differential we can turn the conjugation map

$$G \longrightarrow \mathrm{Aut}(G)$$

$$g \longmapsto g(\cdot)g^{-1}$$

into a map

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g}).$$

This gives a bilinear map

$$\begin{aligned} [\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathfrak{g} \\ [x,y] &\longmapsto \mathrm{ad}(x)y. \end{aligned}$$

We will often drop the comma for brevity.

Example. If $G = \mathrm{GL}_n(\mathbb{R}), \mathfrak{g} = M_{n\times n}(\mathbb{R})$ and $[xy] = xy - yx$

1.2 What are Lie algebras good for?

1. They tell us about the structure of G.

   Example. We'll define the *root system* of $\mathfrak{g}$. This tells you about commutator relations in G (See Carter, Simple Groups of Lie Type).

   Example. We'll define the *Weyl group* of $\mathfrak{g}$. For example, the Weyl group of $\mathrm{GL}_n(\mathbb{C})$ is S_n. There is an embedding $S_n \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ via permutation matrices permuting a basis.

*corrections to ep455@cam.ac.uk

Let B be an upper triangular matrix in G. Then

$$G = \bigsqcup_{w \in S_n} BwB \quad \text{(Bruhat decompostion).}$$

2. They tell us about representation theory.

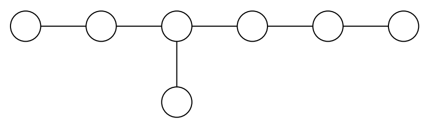
   Example. There's a bijection

   $$\{\text{finite dim representations of} \, \mathrm{SL}_n(\mathbb{C})\} \leftrightarrow \{\text{finite dim representations of } Lie(\mathrm{SL}_n(\mathbb{C}))\}.$$

   We will completely describe the right hand side.

3. They have applications to algebraic geometry. It's possible to use Lie algebras to build families of surfaces or algebraic curves (see Slodowy, Simple Singularities and Simple Algebraic Groups).

   We'll define the *Dynkin diagram* of a semisimple Lie algebra

   Example. 

   This tells you about singularities on surfaces.

4. They have applications to Number Theory. Root systems / Weyl groups give structure of groups over $\mathbb{Q}_p$ (see paper of Iwahori-Matsumoto)

   The local Langlands correspondence predicts a relationship

   $$\{\text{Galois Theory of local fields}\} \leftrightarrow \{\text{Complex Lie Theory}\}$$

There are many other applications that we have not mentioned here (for example to algebraic groups and theoretical physics).

2 Basic Definitions and Examples

Definition. Let k be a field. A *Lie algebra* over k is a vector space $\mathfrak{g}$ over k with a bilinear pairing

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} \quad \text{“Lie Bracket”}$$

such that

1. $[xx] = 0 \ \forall x \in \mathfrak{g}$
2. $[x[yz]] + [y[zx]] + [z[xy]] = 0 \ \forall x, y, z \in \mathfrak{g}$ (The Jacobi Identity).

Exercise. Check that this definition implies that $[xy] = -[yx]$ (Antisymmetry of the bracket).

By definition, $[x+y, x+y] = 0$. So by bilinearity we have

$$0 = [x+y, x+y] = [x+y, x] + [x+y, y] = [xx] + [xy] + [yx] + [yy],$$

so $[xy] = -[yx]$, as by definition we have $[xx] = [yy] = 0$

Definition. $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ if $\mathfrak{h}$ is a subspace of $\mathfrak{g}$ and $[xy] \in \mathfrak{h}$ for all $x, y \in \mathfrak{h}$.

Examples. Let V be a finite dimensional vector space over k.

1. Let $\mathfrak{gl}(V) = \mathrm{End}(V)$ with bracket given by $[xy] = xy - yx$, where the multiplication is in the endomorphism ring. It is clear $[xx] = 0$.

   Exercise. Check the Jacobi identity.

   $[x[yz]] + [y[zx]] + [z[xy]] = x(yz - zy) - (yz - zy)x + y(zx - xz) - (zx - xz)y + z(xy - yx) - (xy - yx)z$. Regrouping, this is $x(yz - zy + zy - yz) + y(-zx + zx - xz + xz) + z(yx - xy + xy - yx) = 0$.

   If we choose a basis for V we can identify $\mathfrak{gl}(V)$ with the space of $n \times n$ matrices over k, so we often write $\mathfrak{gl}(V)$ as $\mathfrak{gl}_n$.

2. Let $\mathfrak{sl}(V) = \{x \in \mathfrak{gl}(V) \mid \mathrm{tr}(x) = 0\}$. This is a subspace, as trace is linear, and closed under the Lie bracket, as trace is symmetric, so it's a subalgebra.

   Note that $\dim(\mathfrak{sl}(V)) = n^2 - 1$. We take the standard basis:

   $$\begin{bmatrix} 1 & & & & \\ & -1 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & & & \\ & 1 & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}, \cdots$$

   $$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ & 0 & & & \\ \vdots & & \ddots & & \\ & & & \ddots & \\ 0 & & & & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & \dots & 0 \\ & 0 & & & \\ \vdots & & \ddots & & \\ & & & \ddots & \\ 0 & & & & 0 \end{bmatrix}, \cdots$$

   and we often write $\mathfrak{sl}_n$.

3. Assume $\mathrm{char}(k) \neq 2$. Suppose V is endowed with a symmetric bilinear form

   $$\langle \cdot, \cdot \rangle : V \times V \longrightarrow k.$$

   Let $\mathfrak{so}(V) = \{x \in \mathfrak{gl}(V) \mid \langle xv, w \rangle = -\langle v, xw \rangle \text{ for all } v, w \in V\}$.

   In co-ordinates, we know that there is a matrix $M \in \mathrm{GL}(V)$ such that $\langle v, w \rangle = v^T M w$, so

   $$\mathfrak{so}(V) = \{x \mid Mx + x^T M = 0\}.$$

We'll usually take

$$M = \begin{cases} \begin{bmatrix} 0 & I_l \\ I_l & 0 \end{bmatrix} & \text{if } n = 2l \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{bmatrix} & \text{if } n = 2l+1. \end{cases}$$

2.0.1 Warm Up for Lecture 2

Let

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

viewed as matrices in $\mathfrak{sl}_2(\mathbb{C})$ - the standard basis from lecture 1. Then we have

$$[ef] = h, \quad [he] = 2e, \quad [hf] = -2f$$

We'll see that in some sense the structure of all semisimple Lie algebras come from $\mathfrak{sl}_2(\mathbb{C})$.

Continuing the examples from last time, and recalling that we had V a k-vector space,

4. Again, assume $\text{char}(K) \neq 2$, and suppose that V is endowed with a non-degenerate skew-symmetric bilinear form $\langle \cdot, \cdot \rangle$ (that is, $\langle v, w \rangle = -\langle w, v \rangle$).

   Exercise. Check 3,4 are Lie subalgebras of $\mathfrak{gl}(V)$.

   Then,

   $$\mathfrak{sp}(V) = \{x \in \mathfrak{gl}(V) \mid \langle xv, w \rangle = -\langle v, xw \rangle \quad \forall v, w \in V\}.$$

   In co-ordinates we'll take $\langle , \rangle$ to be the skew-symmetric form associated to

   $\begin{bmatrix} 0 & I_l \\ -I_l & 0 \end{bmatrix}$ where $n = 2l$. Note n must be even as V has a skew-symmetric form.

5. V is a Lie algebra with bracket $[vw] = 0$ for all v, w.

Note: Our definition of $\mathfrak{gl}(V)$ makes sense for V infinite dimensional.

Definition. A linear transformation $\varphi : g \longrightarrow h$ between two Lie algebras is a *homomorphism* if $[\varphi(x)\varphi(y)] = \varphi([xy])$ for all $x, y \in \mathfrak{g}$. If φ is also an isomorphism of vector spaces it is a *isomorphism*.

3 Representations Part 1

Let $\mathfrak{g}$ be a Lie algebra.

Definition. A *representation* of $\mathfrak{g}$ is a Lie algebra homomorphism $\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ for some V.

Notation.
- We also call V itself a representation.
- We also write $\mathfrak{g} \circlearrowright V$ and say "$\mathfrak{g}$ acts on V."
- We write $x \cdot v$ or xv for $\varphi(x)(v)$.

Definition. The dimension of the representation is $\dim(V)$.

Examples.

1. *(The trivial representation)* Let V be a 1-dimensional vecotr space. Then $\mathfrak{g} \circlearrowright V$ via $x \cdot v = 0$ for all $x \in \mathfrak{g}, v \in V$.
2. *(The defining representation)* If $\mathfrak{g}$ is defined as a subalgebra of $\mathfrak{gl}(V)$ there is a natural inclusion $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$.
3. *(The adjoint representation)* For $x \in \mathfrak{g}$ define $\operatorname{ad} x : \mathfrak{g} \longrightarrow \mathfrak{g}$, $y \longmapsto [xy]$. Then, the map

$$\mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$$
$$x \longmapsto \operatorname{ad} x$$

is the adjoint map.

This is a Lie algebra homomorphism: we'll check $[\operatorname{ad} x, ady](z) = \operatorname{ad}([xy])(z)$. The left hand side is

$$[x[yz]] - [y[xz]] = -[[zx]y] - [[yz]x] = [[xy]z]$$

which is the right hand side, where the last step is an application of the Jacobi identity.

Example. Recalling the multiplications in the warm up, the adjoint representation of $\mathfrak{sl}_2(\mathbb{C})$ (with basis $\{e, h, f\}$ in that order) has

$$\operatorname{ad}(h) = \begin{bmatrix} 2 & & \\ & 0 & \\ & & -2 \end{bmatrix}, \quad \operatorname{ad}(e) = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \operatorname{ad}(f) = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

4. If V, W are representations so is $V \oplus W$ via $x \cdot (v, w) = (xv, xw)$.
5. If V is a representation of $\mathfrak{g}$ then so is the dual V^* via

$$(x \cdot f)(v) = -f(xv)$$

for each $x \in \mathfrak{g}, v \in V, f \in V^*$.

Exercise. Check this is a Lie algebra homomorphism.

6. If V, W are representations of $\mathfrak{g}$ then so is $\mathrm{Hom}(V, W)$ via

$$(x \cdot f)(v) = x \cdot f(v) - f(x \cdot v).$$

Definition. If V, W are representations of $\mathfrak{g}$ then a linear transformation $\varphi : V \longrightarrow W$ is called $\mathfrak{g}$-equivariant if for all $x \in \mathfrak{g}, v \in V$, $x \cdot \varphi(v) = \varphi(x \cdot v)$. V and W are *isomorphic* if there is a $\mathfrak{g}$-equivariant isomorphism $V \longrightarrow W$.

Definition. A *subrepresentation* $V' \subset V$ is a subspace such that $x \cdot v \in V'$ for all $x \in \mathfrak{g}, v \in V'$.

Definition. V is *irreducible* if it has exactly two subrepresentations, namely 0 and V (note the 0 representation is not irreducible).

Examples. 1. The trivial representation is irreducible.

2. For $\mathfrak{sl}_2(\mathbb{C})$, the defining representation and adjoint representation are irreducible.

   Exercise. Prove this.

Definition. V is called *completely reducible* if it decomposes as the direct sum of irreducible representations.

Note: For V a representation, complete reducibility is equivalent to the condition that for every subrepresentation $W \subset V$ there is a W' such that $V = W \bigoplus W'$.

Exercise. Prove this equivalence.

Finally, we have the following example

7. If V is a representation and $W \subset V$ is a subrepresentation then V/W is a representation of $\mathfrak{g}$ via $x(v + W) = xv + W$.

3.0.1 Warm Up for Lecture 3

Let

$$\mathfrak{b} = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in \mathbb{C} \right\}$$

and let $V = \mathrm{span}\left\{ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be the defining representation of $\mathfrak{b}$. Then V is not completely reducible. Suppose that it were. There is a subrepresentation $V_1 = \langle \mathbf{v}_1 \rangle$, since $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$. So, as V is completely reducible there is a subrepresentation V_2 such that $V \cong V_1 \bigoplus V_2$. Suppose $0 \neq \mathbf{v} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 \in V_2$.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_2 \\ 0 \end{pmatrix}$$

so $a_2 = 0$, but then $\mathbf{v} \in V_1$, a contradiction.

Definition. A representation V of a Lie algebra $\mathfrak{g}$ is *faithful* if the map $\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is injective.

From now on in this course, all Lie algebras and representations are over $\mathbb{C}$. Let V be a representation of $\mathfrak{sl}_2(\mathbb{C})$, and let

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We know the following representations already:

dim	name	action of h
1	trivial	0
2	defining	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
3	adjoint	$\begin{bmatrix} 2 & & \\ & 0 & \\ & & -2 \end{bmatrix}$

Definition. For $\lambda \in \mathbb{C}$, the λ-*weight space* of V is

$$V_\lambda = \{v \in V \mid h \cdot v = \lambda v\}.$$

Example. The following are *vector space* sums, not decompositions into irreducible representations:

1. The trivial is V_0.
2. The defining representation is $V_1 \oplus V_{-1}$
3. The adjoint representation $V_2 = \langle e \rangle$, $V_0 = \langle h \rangle$, $V_{-2} = \langle f \rangle$

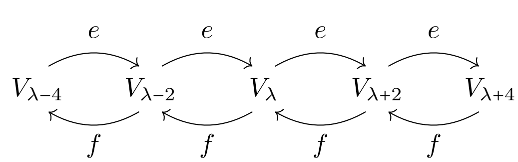
The action of e: Suppose $v \in V_\lambda$. Then

$$h \cdot ev = ([he] + eh) \cdot v = 2e \cdot v + \lambda e \cdot v = (\lambda + 2) e \cdot v, \text{ so } ev \in V_{\lambda+2}.$$

Exercise. $f \cdot v \in V_{\lambda-2}$.

$$h \cdot fv = ([hf] + fh) \cdot v = (-2f + \lambda f) \cdot v = (\lambda - 2) f \cdot v, \text{ so } fv \in V_{\lambda-2}$$

So we have the following picture:



Definition. If a nonzero $v \in V_\lambda \cap \operatorname{Ker}(e)$ for some λ, then v is called a *highest-weight vector* (of weight λ).

Example. In the adjoint representation, e is a highest weight vector.

Lemma 4.1. *Suppose $v \in V_\lambda$ is a highest weight vector. Then for all $n \geq 1$,*

$$ef^n v = n(\lambda - n + 1)f^{n-1}v.$$

Proof. By induction. Base case:

$$efv = ([ef] + fe)v = (h + fe)v = \lambda v(+0) = 1(\lambda - 1 + 1)f^0 v$$

Exercise. Finish the proof.

For $n > 1$, we have

$$ef^n v = ef(f^{n-1}v) = ([ef] + fe)f^{n-1}v = hf^{n-1}v + f(n-1)(\lambda - (n-1) + 1)f^{n-2}v$$

Since $f^{n-1}v \in V_{\lambda - 2n+2}$ this is

$$(\lambda - 2(n-1))f^{n-1}v + (n-1)(\lambda - n + 2)f^{n-1}v = n(\lambda - n + 1)f^{n-1}v$$

which completes the induction, which completes the proof. □

Lemma 4.2. *Suppose $v \in V_\lambda$ is a highest weight vector. Then*

$$W = \operatorname{span}\{v, fv, f^2v, \ldots\}$$

is a subrepresentation of V.

Proof. It suffices to show that if $w = f^n v$ then:

(i) $ew \in W$

(ii) $hw \in W$

(iii) $fw \in W$

(iii) is obvious from the definition of W. For (i), $n \geq 1$ follows from lemma 4.1, and $n = 0$ so $ew = 0$, since v is a highest weight vector. For (ii), $f^n v \in V_{\lambda-2n}$ so $hw = (\lambda - 2n)w \in W$. □

Proposition 4.3. *If V is finite dimensional then it contains a highest weight vector.*

Proof. Choose any nonzero eigenvector v for h, which we can do as we are working over $\mathbb{C}$, etc. Consider

$$v, ev, e^2v, \cdots$$

The set $\{e^n v \mid ev \neq 0\}$ is linearly independent, as the e-action changes the eigenvalue of v, so as V is finite dimensional there is an n such that $e^n v \neq 0$, but $e^k v = 0$ for all $k > n$. Then $e^n v$ is a highest weight vector. □

Lemma 4.4. *Suppose V is finite dimensional and $v \in V_\lambda$ is a highest weight vector. Then $\lambda \in \mathbb{Z}_{\geq 0}$.*

Proof. Any nonzero vectors of the form $f^n v$ must be linearly independent, so there is an $n \geq 0$ such that $f^n v \neq 0$ but $f^k v = 0$ for all $k > n$. Then by Lemma 4.1,

$$0 = ef^{n+1}v = (n+1)(\lambda - n)f^n v.$$

Since $f^n v$ is nonzero, $n = \lambda$. □

Putting this all together, suppose V is of dimension $n+1$ and irreducible. Proposition 4.3 tells us that there is a highest weight vector $v \in V_\lambda$. Lemma 4.2 tells us that $\text{span}\{v, fv, f^2v, \cdots\}$ is a subrepresentation, so $\{v, fv, f^2v, \cdots, f^n v\}$ is a basis, as the $f^i v$ are all linearly independent. The proof of Lemma 4.4 tells us that $\lambda = n$. So we have:

Corollary 4.5. *If V is an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ of dimension $n+1$ then there is a basis $v_0, v_1, \cdots v_n$ of V such that:*

$$\begin{bmatrix} n & & & & \\ & n-2 & & & \\ & & \ddots & & \\ & & & -n-2 & \\ & & & & -n \end{bmatrix} \qquad h \cdot v_i = (n-2i)v_i$$

$$\begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{bmatrix} \qquad f \cdot v_i = \begin{cases} v_{i+1} & 0 \leq i \leq n-1 \\ 0 & i = n \end{cases}$$

$$\begin{bmatrix} 0 & n & & & \\ & 0 & 2(n-1) & & \\ & & \ddots & \ddots & \\ & & & 0 & n \\ & & & & 0 \end{bmatrix} \qquad e \cdot v_i = \begin{cases} i(n-i+1)v_{i-1} & i \geq 1 \\ 0 & i = 0 \end{cases}$$

In particular, there is a unique irreducible representation of dimension $n+1$ for all $n \geq 0$.

Warm Up for Lecture 4: Let V be an $(n+1)$ dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ and let $v \in V$ be a highest weight vector. Then

$$\left(ef + fe + \frac{1}{2}h^2\right)(v) = nv + \frac{n^2}{2}v = \left(\frac{n^2}{2} + n\right)v.$$

Notation. Write $V(n)$ for the irreducible representation of $\mathfrak{sl}_2$ of dimension $n+1$.

Definition. Given a representation V of $\mathfrak{sl}_2$, $\{\lambda \in \mathbb{C} \mid V_\lambda \neq 0\}$ are the *weights* of V.

Today's goal is:

Theorem 4.6. *Every finite dimensional representation of $\mathfrak{sl}_2$ is completely reducible.*

Note: This along with corollary 4.5 implies that the action of h completely determines a finite dimensional representation.

Example. Suppose V is a 5-dimensional representation of $\mathfrak{sl}_2$ and there is a $v \in V$ such that $h \cdot v = 3v$. This implies that the weights contain $3, 1, -1, -3$, so we must have

$$V \cong V(0) \bigoplus V(3)$$

We'll need some general facts: if $\mathfrak{g}$ is an arbitrary Lie algebra and $\varphi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$, and there is some σ commuting with $\varphi(x)$ for all $x \in \mathfrak{g}$. Then:

Fact 1: $\mathrm{Ker}(\sigma - cI_V)$ is a subrepresentation of V for all $c \in \mathbb{C}$

Proof. Exercise □

Fact 2: If V is irreducible, then σ is a scalar.

Proof. There is a $c \in \mathbb{C}$ such that $\mathrm{Ker}(\sigma - cI)$ is nonzero so $\mathrm{Ker}(\sigma - cI)$ is a nonzero subrep, so since V is irreducible, $V = \mathrm{Ker}(\sigma - cI)$. □

Definition. Let V be a subrepresentation of $\mathfrak{sl}_2$. Then

$$\Omega = ef + fe + \frac{1}{2}h^2 \in \mathfrak{gl}(V)$$

is called the *Casimir element.*

Lemma 4.7. *If $\varphi : \mathfrak{sl}_2 \longrightarrow \mathfrak{gl}(V)$ is finite dimensional then Ω commutes with $\varphi(x)$ for all $x \in \mathfrak{sl}_2$.*

Proof. Check that $e\Omega = \Omega e, f\Omega = \Omega f, h\Omega = \Omega h$ (see Grojnowski's notes page 10).
. □

Corollary 4.8. *If V is an irreducible representation of $\mathfrak{sl}_2$, then $\Omega \circlearrowright V$ as a scalar.*

Proof. Schur . □

(Warm up tells us $\Omega \circlearrowright V_n$ by $\frac{n^2}{2} + n$)

Proof. (of Theorem 4.6) Let $\varphi : \mathfrak{sl}_2 \longrightarrow \mathfrak{gl}(V)$ be a finite dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ and suppose $W \subseteq V$ is a suprepresentation. We need to show there is a $U \subseteq V$ so that $V \cong W \oplus U$.

Case 1: W has codimension 1, so $V/W \cong V(0)$.

Case 1A: W is trivial so $\dim V = 2$ and there is a basis of V with respect to which $\mathfrak{sl}_2$ acts on V by $\begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}$. We want to show $V \cong V(0) \oplus V(0)$.

Note that

$$[\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}] = 0 \quad \forall x, y \in \mathbb{C}$$

Since φ respects the bracket,

$$\varphi(h) = [\varphi(e), \varphi(f)] = 0,$$

$$\varphi(e) = \frac{1}{2}[\varphi(h), \varphi(e)] = 0,$$

and

$$\varphi(f) = \frac{1}{2}[\varphi(h), \varphi(f)] = 0.$$

Case 1B: $W \cong V(n)$ is irreducible, $(n > 0)$.

Consider $\Omega \in \mathfrak{gl}(V)$. We will show $V \cong V(n) \oplus \operatorname{Ker}\Omega$. By Schur, the fact that W is irreducible, and the fact that Ω acts on V/W trivially, there is a basis for V such that Ω acts by

$$\begin{bmatrix} c & & & & & & * \\ & c & & & & & * \\ & & c & & & & * \\ & & & c & & & * \\ & & & & c & & * \\ & & & & & c & * \\ & & & & & & 0 \end{bmatrix}.$$

W is nontrivial, so $\operatorname{Ker}(\Omega)$ is nonzero, and it is clear that $W \cap \operatorname{Ker}(\Omega)$ is zero, so $V = W \oplus \operatorname{Ker}(\Omega)$

Case 1C: W arbitrary.

By induction on $\dim V$. The base case is case 1A. Let $W' \subset W$ be a nonzero subrepresentation. $\dim(W/W') < \dim V$ and the codimension of W/W' in V/W' is 1 so by induction this implies

$$V/W' = W/W' \bigoplus W''/W' \quad (\star)$$

for some subspace W'' of V and W''/W' a subprepresentation of V/W'.

$W' \subset W''$ has codimenstion 1 and $\dim W' < \dim V$. Note that W'' is a subrepresentation of V, since W''/W being a representation implies $xw \in W' \subset W''$ for all $x \in \mathfrak{sl}_2$. So, by induction there is a subrepresentation $U \subseteq W''$ such that

$$W'' = W' \oplus U \quad (\star\star)$$

Now we claim $V = W \oplus U$. We know U has dimension 1, and $W \cap U \subseteq W \cap W'' = W'$ since we showed the right hand side of $(\star)$ is a direct sum. So by $(\star\star)$, as the sum is direct, $W \cap U \subset W' \cap U = 0$. Since U has dimension 1 and W codimension 1, we are done by the rank-nullity theorem.

Case 2: Let W be arbitrary.

Consider $\text{Hom}(V, W)$, recalling that $(x \cdot \varphi)(v) = x \cdot (\varphi(v)) - \varphi(x \cdot v)$. Let

$$\mathcal{V} = \{\psi \in \text{Hom}(V, W) : \psi|_W = cI_W \text{ some } c \in \mathbb{C}\}$$

and the subspace

$$\mathcal{W} = \{\psi \in \mathcal{V} : \psi|_W = 0\}.$$

Note:

- The codimension of $\mathcal{W}$ in $\mathcal{V}$ is 1.
- Suppose $\psi|_W = cI_W, x \in \mathfrak{sl}_2, w \in W$. Then

$$(x \cdot \psi)(w) = x \cdot \psi(w) - \psi(x \cdot w) = x(cw) - c(xw) = 0.$$

So, $\mathcal{V}$ is a subrepresentation of $\text{Hom}(V, W)$ and so $\mathcal{W}$ is a subrepresentation of $\mathcal{V}$. By case 1 we may find a one-dimensional subrepresentation $\mathcal{U}$ of $\mathcal{V}$ such that $\mathcal{V} = \mathcal{W} \oplus \mathcal{U}$. Write $\mathcal{U} = \text{span}(g)$ for $g \in \mathcal{V}$, so $g\,|_W = cI_W$ for some nonzero c.

Now claim that as vector spaces, $V = W \oplus \text{Ker}(g)$.

$W \cap \text{Ker}(g) = 0$, so by rank-nullity, $\dim V = \dim W + \dim(\text{Ker}(g))$ as $W = \text{Im}(g)$, so we do have a direct sum of vector spaces.

It remains to show $\text{Ker}(g)$ is a subpresentation of V. Let $v \in \text{Ker}(g)$, $x \in \mathfrak{sl}_2$. Since U is a one-dimensional representation of $\mathfrak{sl}_2$, U is the trivial representation and so $0 = (x \cdot g)(v) = x \cdot g(v) - g(x \cdot v)$ so $x \cdot gv = g(x \cdot v)$ so $0 = g(x \cdot v)$ and we are done.

□

Remark. The main ingredients of the proof were:

1. Existence of Ω.
2. Every one-dimensional representation of $\mathfrak{sl}_2$ is the trivial representation.

4 Tensors

Let V, W be finite dimensional vector spaces, with $\{v_1, \cdots v_k\}, \{w_1, \cdots, w_m\}$ bases for V, W respectively.

Recall $V \otimes W$ has basis $\{v_i \otimes w_j \mid 1 \leq i \leq k, \quad 1 \leq j \leq m\}$ and

1. $c(v \otimes w) = cv \otimes w = v \otimes cw$ for all $v \in V, w \in W, c \in \mathbb{C}$.
2. $(u_1 + u_2) \otimes w = u_1 \otimes w + u_2 \otimes w$ for all $u_1, u_2 \in V, w \in W$.
3. $v \otimes (z_1 + z_2) = v \otimes z_1 + v \otimes z_2$ for all $v \in V, z_1, z_2 \in W$.

If V, W are representations of a Lie algebra $\mathfrak{g}$, then so is $V \otimes W$ with

$$x \cdot (v \otimes w) = (x \cdot v) \otimes w + v \otimes (x \cdot w).$$

Example. If V, W are representations of $\mathfrak{sl}_2$, and $v \in V_\lambda, w \in W_\mu$,

$$h \cdot (v \otimes w) = (\lambda + \mu) v \otimes w.$$

In particular, the weights of $V \otimes W$ are $\{\lambda + \mu : \lambda \text{ weight for } V,\ \mu \text{ weight for } W\}$.

Example. The decomposition of $V(2) \otimes V(2)$

	2	0	−2
2	4	2	0
0	2	0	−2
−2	0	−2	−4

$$V(2) \otimes V(2) \cong V(4) \oplus V(2) \oplus V(0).$$

Definition. The *nth symmetric power* is given by

$$\mathrm{Sym}^n(V) = V \otimes \cdots \otimes V / M_n,$$

where

$$M_n = \mathrm{span}\{u_1 \otimes \cdots \otimes u_n - u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)} \mid \sigma \in S_n, u_i \in V\}.$$

Example. If $n = 2$, $M_2 = \mathrm{span}\{v \otimes u - u \otimes v \mid u, v \in V\}$.

Fact: M_n is a subrepresentation of $V \otimes \cdots \otimes V$ whenever V is a representation of $\mathfrak{g}$, and so $\mathrm{Sym}^n(V)$ is a subrepresentation.

Example. In $\mathrm{Sym}^2(V)$, $v \otimes w = w \otimes v$ so $\mathrm{Sym}^2(V)$ has basis $\{v_i \otimes v_j \mid i \le j\}$. Decomposing $\mathrm{Sym}^2(V(2))$:

$$0 \neq e \otimes e \in \mathrm{Sym}^2(V(2))$$

so $V(4)$ is a subrepresentation, so

$$\mathrm{Sym}^2(V(2)) = V(4) \oplus V(0).$$

Definition. The *nth exterior power* $\bigwedge^n(V) = V \otimes V \otimes \cdots V / N_n$ where $N_n = \mathrm{span}\{u_1 \otimes \cdots \otimes u_n \mid u_i \in V \forall i, u_i = u_j \text{ for some } j\}$.

Example. For $n = 2$, $N_2 = \mathrm{span}\{v \otimes v \mid v \in V\}$, N_n is a subrepresentation of $V \otimes V \cdots \otimes V$. The proof for $n = 2$ is:

Note $(v+u) \otimes (v+u) \in N_2$, $v \otimes v + v \otimes u + u \otimes v + u \otimes u \in N_2$ so $v \otimes u + u \otimes v \in N_2$. So, if $x \in \mathfrak{g}$, $x(v \otimes v) = (xv) \otimes v + v \otimes (xv) \in N_2$.

Notation. The coset of $u_1 \otimes \cdots \otimes u_n$ is denoted $u_1 \wedge \cdots \wedge u_n$. A basis for $\bigwedge^2 V$ is given by $\{u_i \otimes v_j \mid i < j\}$.

Example. Decomposing, $\bigwedge^2 V(2) \cong V(2)$ with basis $\{e \wedge f, e \wedge h, h \wedge f\}$.

5 Semisimple Lie Algebras

Throughout this section $\mathfrak{g}$ is a semisimple Lie algebra over $\mathbb{C}$.

Definition. A subspace $I \subset \mathfrak{g}$ is an *ideal* of $\mathfrak{g}$ is $[xy] \in I$ for all $x \in \mathfrak{g}, y \in I$.

Note that:

1. Any ideal is a subalgebra.
2. If I is an ideal $\mathfrak{g}/I$ is a Lie algebra under $[x+I, y+I] = [xy] + I$.
3. I is an ideal if and only if it is a subrepresentation of the adjoint representation of $\mathfrak{g}$.

Examples. 1. The *center* of $\mathfrak{g}$ is $\mathcal{Z} = \{x \in \mathfrak{g} \mid [xy] = 0 \text{ for all } y \in \mathfrak{g}\}$, which is an ideal.

2. The *derived subalgebra* $[\mathfrak{g}\mathfrak{g}]$ is the span of $\{[xy] \mid x, y \in \mathfrak{g}\}$, and is an ideal.

   Exercise. The derived subalgebra of $\mathfrak{gl}_n$ is $\mathfrak{sl}_n$. This is since $trXtrY = trYtrX$ and the basis elements of $\mathfrak{sl}_n$ generate one another under the bracket.

3. If $\phi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a homomorphism then $\mathrm{Ker}(\phi)$ is an ideal, since ϕ respects the bracket. In fact, every ideal arises in this way.

Definition. If $[\mathfrak{g}\mathfrak{g}]$ is nonzero and the only ideals of $\mathfrak{g}$ are 0 and $\mathfrak{g}$ then $\mathfrak{g}$ is *simple.*

Examples. We'll show $\mathfrak{sl}_n$ for $n \geq 2$, so_n for $n \geq 5$ and sp_{2l} for $l \geq 1$ are all simple. Note:

1. If $\mathfrak{g}$ is simple then $[\mathfrak{g}\mathfrak{g}] = \mathfrak{g}$.
2. If $\mathfrak{g}$ is simple then every representation of $\mathfrak{g}$ is either faithful or a direct sum of trivial representations.

   So, $\mathfrak{g}$ is simple if and only if the adjoint representation is irreducible.

Definition. The Lie algebra $\mathfrak{g}$ is *semisimple* if it is the direct sum of *simple ideals*, i.e. ideals which are themselves simple when viewed as Lie algebras.

Example. $so_2 \cong sl_2 \oplus sl_2$.

Non-example. $\mathfrak{gl}_n \cong \oplus sl_n$.

Our aim will be to state a more standard definition of a semisimple Lie algebra and then to show this definition is equivalent.

Definition. The *central series* $\mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \mathfrak{g}^2 \cdots$ is given by $\mathfrak{g}^0 = \mathfrak{g}, \mathfrak{g}^n = [\mathfrak{g}, \mathfrak{g}^{n-1}]$. That is,

$$\mathfrak{g} \supset [\mathfrak{g}\mathfrak{g}] \supset [\mathfrak{g}[\mathfrak{g}\mathfrak{g}]] \supset \cdots.$$

Definition. The *derived series* $\mathfrak{g}^{(0)} \supset \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \cdots$ is given by $\mathfrak{g}^{(0)} = \mathfrak{g}, \mathfrak{g} = [\mathfrak{g}^{(n-1)}\mathfrak{g}^{(n-1)}]$ for each $n \geq 1$. That is,

$$\mathfrak{g} \supset [\mathfrak{g}\mathfrak{g}] \supset [[\mathfrak{g}\mathfrak{g}][\mathfrak{g}\mathfrak{g}]] \supset \cdots.$$

Note that:

1. $\mathfrak{g}^{(n)} \subset \mathfrak{g}^n$.
2. For all n, $\mathfrak{g}^{(n)}, \mathfrak{g}^n$ are ideals.

   Exercise. Prove this.

   By induction on n. When $n = 0$ we're done so let $n > 0$. Let $x, y \in \mathfrak{g}, \mathfrak{g}^n$ respectively. Then $[xy] \in \mathfrak{g}^{n-1}$ since $\mathfrak{g}^{n-1}$ is an ideal, and $\mathfrak{g}^n \subset \mathfrak{g}^{n-1}$. But $\mathfrak{g}^n$ *is* $\{[xy] : x \in \mathfrak{g}, y \in \mathfrak{g}^{n-1}\} \supset \{[xy : x \in \mathfrak{g}, y \in \mathfrak{g}^n]\}$.

   Similarly, $\mathfrak{g}^{(n)} = \{[xy] : x, y \in \mathfrak{g}^{(n-1)}\}$ and so if $y \in \mathfrak{g}^{(n)}$, $y = [wz], w, z \in \mathfrak{g}^{(n-1)}$ so for $x \in \mathfrak{g}$, $[xy] = [x[wz]] = -[w[xz]] - [z[xw]]$. By induction, $[zx]$ and $[xw]$ are in $\mathfrak{g}^{(n-1)}$ so $[xy] \in \mathfrak{g}^{(n)}$ by definition.
3. If $\mathfrak{g}$ is simple, both series look like $\mathfrak{g} \supset \mathfrak{g} \supset \cdots$.
4. If $\mathfrak{g}$ is abelian, both series look like $\mathfrak{g} \supset 0 \supset 0 \supset \cdots$.

Example. Let

$$\eta = \left\{ \begin{bmatrix} 0 & * & * & * \\ & \ddots & * & * \\ & & \ddots & * \\ & & & 0 \end{bmatrix} \right\} \subset \mathfrak{gl}_n.$$

The central series for η is

$$\left\{ \begin{bmatrix} 0 & * & * & * \\ & \ddots & * & * \\ & & \ddots & * \\ & & & 0 \end{bmatrix} \right\} \supset \left\{ \begin{bmatrix} 0 & 0 & * & * \\ & \ddots & \ddots & * \\ & & \ddots & 0 \\ & & & 0 \end{bmatrix} \right\} \supset \left\{ \begin{bmatrix} 0 & 0 & 0 & * \\ & \ddots & \ddots & 0 \\ & & \ddots & 0 \\ & & & 0 \end{bmatrix} \right\} \supset \cdots.$$

Definition. If $\mathfrak{g}^n = 0$ for some n then $\mathfrak{g}$ is called *nilpotent*.

Definition. If $\mathfrak{g}^{(n)} = 0$ for some n then $\mathfrak{g}$ is called *solvable*.

Example. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} * & * & * & * \\ & \ddots & * & * \\ & & \ddots & * \\ & & & * \end{bmatrix} \right\} \subset \mathfrak{gl}_n.$$

Exercise. $\mathcal{B}$ is solvable but not nilpotent.

Abuse notation and write h for the image of h in $\mathfrak{gl}_n$ under the irreducible representation of $\mathfrak{sl}_2$ of dimension n and analogously for e, both with respect to the basis for which they take the standard form. Then since $h \in \mathcal{B}$, and $[he] = 2e$, by induction $e \in \mathcal{B}^n$ for all n and so $\mathcal{B}$ is not nilpotent.

To see that $\mathcal{B}$ is solvable, note that the diagonal of the bracket of any two elements of $\mathcal{B}$ is identically zero. So, the derived series of $\mathcal{B}$ after the zeroth term is contained in the central series of η and therefore becomes zero after n steps.

Theorem. (Lie's Theorem) If $\mathfrak{g}$ is a solvable subalgebra of $\mathfrak{gl}(V)$ where V is a finite dimensional $\mathbb{C}$-vector space then there is a basis for V such that every element is upper triangular.

Proof. See Humphreys, Section 4. □

Proposition 6.1. *Suppose I, J are ideals of $\mathfrak{g}$.*

1. *If $\mathfrak{g}$ is solvable, then any subalgebra or quotient of $\mathfrak{g}$ is solvable.*
2. *If I is solvable and $\mathfrak{g}/I$ is also solvable then so is $\mathfrak{g}$.*
3. *If I, J are both solvable then so is $I + J$.*

Proof.

1. Follows immediately from the definition.
2. Choose n such that $(\mathfrak{g}/I)^{(n)} = 0$, so $\mathfrak{g}^{(n)} \subset I$. Note $\mathfrak{g}^{(n+m)} \subset I^{(m)}$ for each $m \geq 0$, and since I is solvable we're done.
3. We have $(I + J)/J \cong I/I \cap J$. The right hand side is solvable by 1. J is solvable by assumption and $(I+J)/J$ is solvable. So by 2, $I+J$ is solvable. □

Proposition 6.1.3 allows us to define the radical of $\mathfrak{g}$.

Definition. The *radical* of $\mathfrak{g}$ is the maximal solvable ideal of $\mathfrak{g}$, denoted by $\mathrm{Rad}(\mathfrak{g})$.

Definition. If $\varphi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a finite dimensional representation of $\mathfrak{g}$ the *trace form* of V is

$$\begin{aligned}(\cdot,\cdot)_V : \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathbb{C}\\ (x, y) &\longmapsto \mathrm{tr}(\varphi(x)\varphi(y)).\end{aligned}$$

Exercise.

1. Trace forms are symmetric bilinear forms. It is clear that they are biilinear forms, and symmetry follows since $\mathrm{tr}(XY) = \mathrm{tr}(YX)$ and trace is independent of basis.
2. $([xy]z)_V = (x[yz])_V$.

Definition. The *Killing form* $K(\cdot,\cdot)$ is $(\cdot,\cdot)_{\mathrm{ad}}$, i.e. $K(x,y) = \mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y))$.

Theorem 6.2. *The following are equivalent for a finite dimensional Lie algebra* $\mathfrak{g}$*:*

1. $\mathfrak{g}$ *is semisimple.*
2. $\mathrm{Rad}(\mathfrak{g}) = 0$.
3. *The Killing form on* $\mathfrak{g}$ *is non-degenerate.*

Lemma 6.3. *Let* $\mathfrak{g}$ *be a Lie algebra.*

1. *If* I *is an ideal of* $\mathfrak{g}$ *then so is* $[II]$.
2. $\mathrm{Rad}(\mathfrak{g}) = 0$ *if and only if* $\mathfrak{g}$ *has no nontrivial abelian ideals.*

Proof. For the first part, if $x, y \in I$, and $z \in \mathfrak{g}$ just need to show that $[z[xy]] \in [II]$. By the Jacobi identity,

$$[z[xy]] = -[x[yz]] - [y[zx]].$$

For both of the summands on the right, both components of the bracket are in I since I is an ideal, so the left hand side is in $[II]$ as required.

For the second part, it is immediate that any abelian ideal is solvable. If I is solvable, the last nonzero term in the derived series for I is abelian. □

Warm Up for Lecture 7: Define $\mathfrak{g}^{\perp} = \{x \in \mathfrak{g} \mid K(x,y) = 0 \forall y \in \mathfrak{g}\}$. Claim that $\mathfrak{g}^{\perp}$ is an ideal. Let $x \in \mathfrak{g}^{\perp}, y, z \in \mathfrak{g}$. We are required to show that $K([xy], z) = 0$. We know that $K([xy], z) = K(x, [yz]) = 0$ since $s \in \mathfrak{g}^{\perp}$ and we're done.

In order to prove Theorem 6.2, we need a few results to start us off. The key ingredients are:

Lemma 6.4. *Let* I *be an ideal of* $\mathfrak{g}$ *and let* K_I *be the Killing form of* I*. Then* $K_I(xy) = K(xy)$ *for all* $x, y \in I$.

Proof. By multiplying matrices. Choose a basis of I and extend to a basis of $\mathfrak{g}$. Let $x, y \in I$ With respect to this basis:

$$\mathrm{ad}(x) = \begin{bmatrix} A & * \\ 0 & 0 \end{bmatrix} \text{ where } A = (\mathrm{ad}\, x)\,|_I$$

and similarly for $\mathrm{ad}(y)$. $K_I(x,y) = \mathrm{tr}(AB) = \mathrm{tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y) = K(x,y)$. □

Cartan's Criterion Suppose $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(V)$ for V a finite dimensional $\mathbb{C}$-space. If $(x,y) = 0$ for all $x \in \mathfrak{g}$, $y \in [\mathfrak{g}\mathfrak{g}]$, then $\mathfrak{g}$ is solvable.

Proof. See Humphrey's 4.3, uses Jordan decomposition. □

Corollary 6.5. *1. If $\mathfrak{g} = \mathfrak{g}^\perp$ then $\mathfrak{g}$ is solvable.*

2. If $\mathfrak{g}$ is simple, then $\mathfrak{g}^\perp = 0$.

3. $\mathfrak{g}^\perp$ is solvable for all finite dimensional $\mathfrak{g}$.

Proof. 1. Consider the adjoint $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. The image $\mathrm{ad}(\mathfrak{g}) \cong \mathfrak{g}/\mathcal{Z}(\mathfrak{g})$. $\mathcal{Z}(\mathfrak{g})$ is solvable since it is abelian. Since $\mathfrak{g} = \mathfrak{g}^\perp$, Cartan's criterion implies that $\mathrm{ad}(\mathfrak{g})$ is solvable. So by Proposition 6.1, $\mathfrak{g}$ is solvable.

2. $\mathfrak{g}\perp$ is an ideal so either $\mathfrak{g}^\perp = 0$ in which case we're done or $\mathfrak{g}^\perp = \mathfrak{g}$, but then $\mathfrak{g}$ is solvable by (1), which contradicts the fact that $\mathfrak{g}$ is simple, and has derived subalgebra identically $\mathfrak{g}$.

3. $(\mathfrak{g}^\perp)^\perp = \mathfrak{g}^\perp$ by Lemma 6.4, so by (1) $\mathfrak{g}^\perp$ is solvable.

□

Proof. (of theorem) To see that (2) implies (3), $\mathfrak{g}$ is a solvable ideal so $\mathfrak{g}^\perp \subset \mathrm{Rad}(\mathfrak{g}) = 0$

To see that (3) implies (2), let A be an abelian ideal of $\mathfrak{g}$. We claim that $A \subset \mathfrak{g}^\perp$. Let $x \in A$, $y \in \mathfrak{g}$. Choose a basis for A and extend to $\mathfrak{g}$

$$\mathrm{ad}(x) = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \quad \mathrm{ad}(y) = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$$

so $\mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)) = 0$ so $A = 0$.

To see that (2),(3) imply (1), note that if $\mathfrak{g}$ is simple then we are done and if not we can choose a minimal nontrivial ideal. Let

$$\mathfrak{g}_I = \{x \in \mathfrak{g} \mid K(x,y) = 0 \forall y \in I\}$$

This is an ideal of $\mathfrak{g}$ - the proof is the same as in the warm up.

Now, claim that $\mathfrak{g} = I \oplus \mathfrak{g}_I$.

To see this, since I is simple by minimality (and being nonabelian by (2)), $I \cap \mathfrak{g}^I \subset I^\perp = 0$. Now consider the map

$$\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^* \xrightarrow{res} I^*$$

$$x \longrightarrow K(x,\cdot) \longrightarrow K|_{I^*}(x,\cdot).$$

The kernel is $\mathfrak{g}_I$, which proves claim 1.

Repeat the argument with $\mathfrak{g}_I$ (choose a minimal ideal of $\mathfrak{g}_I$). We can do this because:

Exercise. Any ideal of $\mathfrak{g}_I$ is an ideal of $\mathfrak{g}$ and so $\mathrm{Rad}(\mathfrak{g}_I) = 0$

Claim 2: $(\mathfrak{g}_I)^\perp = 0$, since if $x \in (\mathfrak{g}_I)^\perp$ then $x \in \mathfrak{g}^\perp$.

To see that (1) implies (2), write $\mathfrak{g} = \oplus I_i$, I_i simple ideals. Let p_i be the projection onto I_i.

Exercise. If J is an ideal of $\mathfrak{g}$ then $p_i(J)$ is an ideal of I_i.

If $A \subset \mathfrak{g}$ is an abelian ideal of $\mathfrak{g}$ then $p_i(A)$ is an abelian ideal of I_i so $p_i(A) = 0$ for all i, and therefore $A = 0$. □

Theorem. (Weyl's Theorem) Any finite dimensional representation of a semisimple Lie algebra is completely reducible.

Proof. Almost the same as for $\mathfrak{sl}_2$, the main ingredient being the second version of the Casimir element. □

Exercise. Any ideal or quotient of a semisimple Lie algebra is also semisimple.

Now, let $\varphi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a finite dimensional irreducible representation. Without loss of generality assume φ is faithful, otherwise could work with $\mathfrak{g}/\operatorname{Ker}(\varphi)$. By Cartan's criterion, $(\cdot,\cdot)_V$ is non-degenerate. Choose a basis $x_1, \cdots, x_n$ of $\mathfrak{g}$. Since the trace form is non-degenerate pick dual basis $y_1, \cdots, y_n$. Let

$$\Omega_\varphi = \sum_i \varphi(x_i)\varphi(y_i).$$

Then Ω_φ commutes with $\varphi(x)$ for all $x \in \mathfrak{g}$ (Humphrey's 6.2). By Schur, Ω_φ is a scalar. But $\operatorname{tr}(\Omega_\varphi) = \sum_i \operatorname{tr}(\varphi(x)\varphi(y) = \dim(\mathfrak{g})$ so $\Omega_\varphi = 0$.

Warm up for lecture 8: If $\mathfrak{g}$ is a simple Lie algebra, $\varphi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a finite dimensional representation, then $\varphi((g)) \subset \mathfrak{sl}(V)$. This is because $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$ so

$$\varphi(\mathfrak{g}) = \varphi([\mathfrak{g}\mathfrak{g}]) = [\varphi(\mathfrak{g}), \varphi(\mathfrak{g})] \subset [\mathfrak{gl}(V), \mathfrak{gl}(V)] = \mathfrak{sl}(V).$$

6 Jordan Decomposition

Recall from linear algebra that if $x \in \mathfrak{gl}(V)$ there is a basis of V such that x is block diagonal with blocks of the form

$$\begin{bmatrix} \lambda & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & 1 & \\ & & & \lambda & \end{bmatrix}$$

Definition. x is *nilpotent* if $x^n = 0$ for some n, and *semisimple* if if the roots of its minimal polynomial are distinct (i.e. it is diagonalisable).

Proposition 7.1. *If $x \in \mathfrak{gl}(V)$ then:*

1. *There are unique x_s, x_n in $\mathfrak{gl}(V)$ with x_s semisimple, x_n nilpotent, $x = x_s + x_n$, and $[x_s, x_n] = 0$.*

2. *There are polynomials* p_s, p_n *in* $\mathbb{C}[T]$ *such that* $p_s(0) = p_n(0) = 0$ *and* $p_s(x) = x_s, \quad p_n(x) = x_n.$ □

Definition. x_s is the *semisimple part* of x, x_n is the *nilpotent part.*

Suppose $\mathfrak{g}$ is a finite dimensional Lie algebra, then we have the map

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$$

so $\mathrm{ad}(x)$ has a Jordan decomposition for all $x \in \mathfrak{g}$.

Lemma 7.2. *Suppose* $\mathfrak{g}$ *is a subalgebra of* $\mathfrak{gl}(V)$*, and* $x \in \mathfrak{g}$*. Then* $\mathrm{ad}\, x = \mathrm{ad}\, x_s + \mathrm{ad}\, x_n$ *is the Jordan decomposition of* $\mathrm{ad}\, x$ *in* $\mathfrak{g}$*.*

Proof. We know that $\mathrm{ad}\, x_s$ is semisimple, and $\mathrm{ad}\, x_n$ is nilpotent, and $[\mathrm{ad}\, x_s, \mathrm{ad}\, x_n] = \mathrm{ad}([x_s, x_n]) = 0$ so by uniqueness the Jordan decomposition must be $\mathrm{ad}\, x = \mathrm{ad}\, x_s + \mathrm{ad}\, x_n$. □

Theorem 7.3. *Suppose* $\mathfrak{g}$ *is semisimple, and a subalgebra of* $\mathfrak{gl}(V)$*, with* $x \in \mathfrak{g}$*. Then* $x_s, x_n \in \mathfrak{g}$*.*

Proof. Let

$$N(\mathfrak{g}) = \{y \in \mathfrak{g} \mid \; [yz] \in \mathfrak{g} \; \forall z \in \mathfrak{g}\}.$$

Then we claim

1. $N(\mathfrak{g})$ is a subalgebra of $\mathfrak{gl}(V)$.
2. $\mathfrak{g}$ is an ideal of $N(\mathfrak{g})$.
3. $x_s, x_n \in N(\mathfrak{g})$..

The first two are clear from the definition of $N(\mathfrak{g})$. For the third, for $z \in \mathfrak{g}$, we have

$$[x_s, z] = \mathrm{ad}\, x_s(z) = (\mathrm{ad}\, x)_s(z),$$

and by proposition 7.1 (2), this is in $\mathfrak{g}$ since $(\mathrm{ad}\, x)_s$ is a polynomial in $\mathrm{ad}\, x$ with no constant term.

Given a subrepresentation $W \subseteq V$ let

$$\mathfrak{g}_W = \{y \in \mathfrak{gl}(V) \; \mid \; yw \in W \;\; \forall w \in W \text{ and } \operatorname{tr}(y\,|_W) = 0\}.$$

Then $\mathfrak{g}_W$ is a subalgebra of $\mathfrak{gl}(V)$, by the warm up for lexture 8, $\mathfrak{g} \subseteq \mathfrak{g}_W$ and $x_s, x_n \in \mathfrak{g}_W$. Now, let

$$\mathfrak{g}' = \bigcap_{W \subseteq V,\ \mathfrak{g} \text{ subreps}} \mathfrak{g}_W \cap N(\mathfrak{g}) \supseteq \mathfrak{g}.$$

We now claim $\mathfrak{g} = \mathfrak{g}'$. To see this, $\mathfrak{g}'$ is a representation of $\mathfrak{g}$ via restriction to the adjoint representation, and $\mathfrak{g}$ is a subrepresentation of this representation. By Weyl, $\mathfrak{g}' = \mathfrak{g} \oplus U$ for some representation U, and it suffices to show $U = 0$.

We know that $V = \bigoplus_i V^i$ for irreducible representations V^i, and we'll show that U acts on each V^i by zero. Suppose $u \in U$. Since $\mathfrak{g}$ is an ideal of $\mathfrak{g}'$, if $y \in \mathfrak{g}$ then

$$[y, u] \in \mathfrak{g} \cap U = 0.$$

So u commutes with everything in $\mathfrak{g}$, so by Schur u acts as a scalar on V^i. u also has trace 0 by the definition of $\mathfrak{g}_W$, but $u|_{V^i}$ has trace 0, so $u = 0$. □

The upshot: Suppose $\mathfrak{g}$ is a semisimple finite dimensional Lie algebra. Then $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(g)$ is injective, so given $x \in \mathfrak{g}$,

$$\mathrm{ad}\, x = (\mathrm{ad}\, x)_s + (\mathrm{ad}\, x)_n \quad \text{(Jordan decomposition)}$$

so $(\mathrm{ad}\, x)_s, (\mathrm{ad}\, x)_n \in \mathrm{ad}(\mathfrak{g})$ so we can uniquely define the semisimple and nilpotent parts of x to be x_s, x_n such that

$$\mathrm{ad}(x_s) = (\mathrm{ad}\, x)_s, \;\; \mathrm{ad}(x_n) = (\mathrm{ad}\, x)_n.$$

Proof. Suppose $\varphi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a finite dimensional representation, $x \in \mathfrak{g}$. Then

$$\varphi(x) = \varphi(x_s) + \varphi(x_n)$$

is the Jordan decomposition of $\varphi(x)$ is $\mathfrak{gl}(V)$. □

7 A Brief Introduction to Inner Automorphisms

Let V be a finite-dimensional vector space over $\mathbb{C}$.

Definition. If $x \in \mathfrak{gl}(V)$,

$$\exp(x) = \sum_0^\infty \frac{x^i}{i!}.$$

Note that $\exp(x)$ is invertible, since we can choose a basis such that x is upper triangular.

Lemma 8.1. *If $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(V)$ and $x \in \mathfrak{g}$ is nilpotent then*

$$\exp(\mathrm{ad}\, x)(y) = exp(x) y exp(x)^{-1}.$$

□

Corollary 8.2. *If $\mathfrak{g} \in \mathfrak{gl}(V)$ and x is nilpotent then* $\exp(\mathrm{ad}(x))$ *is an automorphism of $\mathfrak{g}$.* □

Definition. Let G_{ad} be the subgroup of the automorphism group of $\mathfrak{g}$ generated by $\{exp(\mathrm{ad}(x)) \mid x \text{ is nilpotent}\}$ Then G_{ad} is the *group of inner automorphisms of* $\mathfrak{g}$.

Examples. 1. $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, $G_{\mathrm{ad}} = \mathrm{PGL}_n(\mathbb{C})$.

2. $\mathfrak{g} = \mathfrak{so}_n(\mathbb{C})$, $G_{\mathrm{ad}} = \mathrm{SO}_n(\mathbb{C})/\mathcal{Z}$.

3. $\mathfrak{g} = \mathfrak{sp}_{2l}(\mathbb{C})$, $G_{\mathrm{ad}} = \mathrm{Sp}_{2l}(\mathbb{C})/\mathcal{Z}$.

Root Space Decomposition

Throughout, $\mathfrak{g}$ is a finite semisimple Lie algebra over $\mathbb{C}$.

Definition. A subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$ is *toral* if

1. $\mathfrak{t}$ is abelian;
2. $\operatorname{ad} x$ is semisimple for all $x \in \mathfrak{t}$.

A maximal toral subalgebra is called a *Cartan subalgebra* (CSA).

Warning: This is not the standard definition, but is equivalent.

Example. For $\mathfrak{sl}_n, \mathfrak{so}_n, \mathfrak{sp}_{2l}$, with the bilinear form we chose earlier, the space of diagonal matrices forms a Cartan subalgebra.

Lemma 9.1. *Suppose V is a finite dimension $\mathbb{C}$-space and $\sigma_1, \cdots, \sigma_n$ are commuting semisimple endomorphisms of V. Given $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$, define*

$$V_\lambda = \{v \in V \mid \sigma_i(v) = \lambda_i(v) \text{ for all } i\}.$$

Then

$$V = \bigoplus_{\lambda \in \mathbb{C}^n} V_\lambda$$

Proof. The proof is by induction on n. For the $n = 1$ case, this follows since σ_1 is semisimple, so diagonalisability is that V has a basis of σ_1 eigenvectors.

For $n > 1$, we know by the inductive hypothesis that

$$V = \bigoplus_{\lambda' \in \mathbb{C}^{n-1}} V_{\lambda'}$$

for the action of $\sigma_1, \cdots, \sigma_{n-1}$. Since the σ_i commute, $\sigma_n(V_{\lambda'}) = V_{\lambda'}$ for all λ' so decomposing each $V_{\lambda'}$ for σ_n as in the $n = 1$ case, we are done. □

Lemma 9.2. *Any $\mathfrak{g}$ contains a Cartan subalgebra.*

Proof sketch: By Engel's Theorem, we can choose x not nilpotent, and x_s generates a toral subalgebra. Then use Zorn's Lemma.

Rewriting Lemma 9.1, suppose $\mathfrak{h} \subseteq \mathfrak{gl}(V)$ with a basis of commuting semisimple $\sigma_1, \cdots \sigma_n$, $\lambda \in \mathbb{C}^n$ corresponds to the element of $\mathfrak{h}^*$ given by $\sigma_i \longmapsto \lambda_i$. Then

$$V_\lambda = \{v \in V \mid h \cdot v = \lambda(h) \cdot v \ \forall h \in \mathfrak{h}\}.$$

In our situation: fix $\mathfrak{t} \subseteq \mathfrak{g}$ a Cartan subalgebra. Then

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{t}^*} \mathfrak{g}_\lambda$$

where

$$\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid [tx] = \lambda(t)x \text{ for all } t \in \mathfrak{t}\}.$$

Definition. Let $\Phi = \{\alpha \in \mathfrak{t}^* \backslash 0 \mid \mathfrak{g}_\alpha \neq 0\}$. The elements of Φ are the *roots* of $\mathfrak{g}$, with respect to $\mathfrak{t}$. If $\alpha \in \Phi$, $\mathfrak{g}_\alpha$ is a *root space*, and

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

is the *root space decomposition* or *Cartan decomposition* of $\mathfrak{g}$.

Proposition 9.3. *1. For all $\alpha, \beta \in \mathfrak{t}^*$, $[\mathfrak{g}_\alpha \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.*

2. If $\alpha \in \Phi$, then if $x \in \mathfrak{g}_\alpha$, $\operatorname{ad} x$ is nilpotent.

3. If $\alpha + \beta \neq 0$, $K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for all $\alpha, \beta \in \mathfrak{t}^*$.

Proof. 1. Let $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$, and $t \in \mathfrak{t}$. We have

$$[t[xy]] = -[x[yt]] - [y[tx]] = [x[ty]] - \alpha(t)[yx] = \beta(t)[xy] + \alpha(t)[xy].$$

2. follows from (1) and the finite dimensionality of $\mathfrak{g}$.

3. If $\alpha + \beta \neq 0$ then there is a $t \in \mathfrak{t}$ such that $(\alpha + \beta)(t) \neq 0$, so fix such a t, and fix $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$. Then

$$\alpha(t)K(x,y) = K([tx],y) = -K([xt],y) = -K(x,[ty]) = -\beta(t)K(x,y)$$

so $(\alpha + \beta)(t)K(x,y) = 0$, so $K(x,y) = 0$.

□

Corollary 9.4. *1. $K|_{\mathfrak{g}_0 \times \mathfrak{g}_0}$ is non-degenerate.*

2. If $\alpha \in \Phi, -\alpha \in \Phi$.

Proof. Since K is non-degenerate, given a non-zero $x \in \mathfrak{g}_\alpha$, there must be a $y \in \mathfrak{g}_{-\alpha}$ such that $K(x,y) \neq 0$. □

Proposition 9.5. $\mathfrak{g}_0 = \mathfrak{t}$.

Proof. c.f. Humphreys 8.2. □

Corollary 9.6. *$K|_{\mathfrak{t} \times \mathfrak{t}}$ is non-degenerate.*

In particular, the map

$$\mathfrak{t} \longrightarrow \mathfrak{t}^*$$
$$x \longmapsto K(x, \cdot)$$

is an isomorphism. Let the inverse be $\lambda \mapsto t_\lambda$, so t_λ is an element of $\mathfrak{t}$ defined by $K(t_\lambda, x) = \lambda(x)$ for all $x \in \mathfrak{t}$.

Examples. 1. For $\mathfrak{sl}_2$, can take $\mathfrak{t} = \left\langle \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \right\rangle$. Define $\alpha \in \mathfrak{t}^*$ by $\alpha(h) = 2$, $\mathfrak{g}_\alpha = \langle e \rangle, \mathfrak{g}_{-\alpha} = \langle f \rangle$.

2. Let $\mathfrak{g} = \mathfrak{sl}_3$,

$$h_1 = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}, \quad h_2 = \begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix}$$

and let $\mathfrak{t} = \langle \{h_1, h_2\} \rangle$. Define $\alpha \in \mathfrak{t}^*$ by $\alpha(h_1) = 2, \alpha(h_2) = -1$. Then

$$\mathfrak{g}_\alpha = \begin{bmatrix} & 1 & \\ & & \\ & & \end{bmatrix}, \quad \mathfrak{g}_{-\alpha} = \begin{bmatrix} & & \\ 1 & & \\ & & \end{bmatrix}.$$

Theorem 9.7. *If $\alpha \in \Phi$, then*

$$m_\alpha := \mathfrak{g}_\alpha \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_{-\alpha}$$

is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$. In particular,

$$\dim(\mathfrak{g}_\alpha) = \dim([\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]) = \dim(\mathfrak{g}_{-\alpha}) = 1.$$

Stepping back, this means that every semisimple Lie algebra is built out of $\mathfrak{sl}_2$s. To prove this, we'll need some preparation.

Warm Up for Lecture 10: If $t \in \mathfrak{t}$ satisfies $\alpha(t) = 0$ for all $\alpha \in \Phi$, then $t = 0$. To see this, if $\alpha \in \Phi$ and $x \in \mathfrak{g}_\alpha$ then $0 = \alpha(t)x = [tx]$, and since a toral subalgebra is abelian, this holds in all of $\mathfrak{g}$, so $t \in \mathcal{Z}(\mathfrak{g}) = \{0\}$.

Proposition 9.8. *Φ spans $\mathfrak{t}^*$.*

Proof. If not then there is a nonzero t satisfying $\alpha(t) = 0$ for all $\alpha \in \Phi$, which is false by the warm up. □

Proof. (of Theorem 9.7)

Claim 1: $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is one dimensional.

Proof of Claim 1: Suppose $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$ so $[xy] \in \mathfrak{t}$. Let $t \in \mathfrak{t}$, so

$$K([xy], t) = K(x, [yt]) = -K(x, [ty]) = \alpha(t)K(x, y).$$

So, $[xy] = K(x, y)t_\alpha \in \langle t_\alpha \rangle$, so $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is at most one-dimensional, but there are x, y such that $K(x, y) \neq 0$ by non-degeneracy so it is exactly one-dimensional. □

Claim 2: $\alpha(t_\alpha) \neq 0$.

Proof of Claim 2: Since the Killing form is non-degenerate and rescaling is possible, we may choose $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$ such that $K(x,y) = 1$. Then

$$[x,y] = t_\alpha \quad [t_\alpha, x] = \alpha(t_\alpha)x \quad [t_\alpha, y] = -\alpha(t_\alpha)y$$

and therefore $\langle\{x, y, t_\alpha\}\rangle$ is a subalgebra, $\mathfrak{h}$, say, of $\mathfrak{g}$.

Suppose that $\alpha(t_\alpha) = 0$. Since $[\mathfrak{h}, \mathfrak{h}] = \langle t_\alpha \rangle$, this implies that $\mathfrak{h}$ is solvable. Consider $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$. We know that $\mathfrak{h}$ embeds to a solvable subalgebra, so by Lie's Theorem there is a basis of $\mathfrak{gl}(\mathfrak{g})$ with respect to which $\mathrm{ad}(h)$ is contained in the set of upper triangular matrices, and $\mathrm{ad}(\mathfrak{t}_\alpha) = [\mathrm{ad}\, x, ady]$ is a strictly upper triangular matrix. So $\mathrm{ad}(t_\alpha)$ is nilpotent, but by semisimplicity it is semisimple and therefore $\mathrm{ad}(t_\alpha) = 0$ so $t_\alpha \in Z(\mathfrak{g}) = 0$, a contradiction. □ .

Notation: Given $\alpha \in \Phi$, write $h_\alpha = \frac{2t_\alpha}{\alpha(t_\alpha)} \in \mathfrak{t}$. Choose $e_\alpha \in \mathfrak{g}_\alpha$, $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $[e_\alpha, e_{-\alpha}] = h_\alpha$. Then note that

$$[h_\alpha, e_\alpha] = \alpha(h_\alpha)e_\alpha = 2e_\alpha, \quad [h_\alpha, e_{-\alpha}] = -2e_{-\alpha}$$

and therefore $s_\alpha := \langle\{e_\alpha, h_\alpha, e_{-\alpha}\}\rangle \cong \mathfrak{sl}_2$ with $(e_\alpha, h_\alpha, e_{-\alpha})$ an $\mathfrak{sl}_2$ triple.

For the rest of the proof, let

$$V = \mathfrak{t} \oplus \sum_{c \in \mathbb{C}} \mathfrak{g}_{c\alpha}.$$

Then, via restriction of the adjoint map V is a representation of s_α, and we may note that:

1. $\mathfrak{t} = \langle \mathfrak{h}_\alpha \rangle \oplus \mathrm{Ker}(\alpha)$.
2. $\mathrm{Ker}(\alpha)$ is an s_α subrepresentation of V.
3. s_α is an s_α subrepresentation of V.

By Weyl's theorem, as a representation of s_α,

$$V = \mathrm{Ker}(\alpha) \oplus s_\alpha \oplus W \text{ (for some complement } W).$$

Claim 3 : $W = 0$.

Proof of Claim 3: It suffices to show that any irreducible representation of W is zero. Let W_0 be a nonzero irreducible representation. Then W_0 has a highest weight vector, w_0, and we know:

- $w_0 \in \mathfrak{g}_{c\alpha}$ for some $c \neq 0$.
- $[h_\alpha, w_0] = nw_0$ some $n \in \mathbb{Z}_{\geq 0}$
- $[h_\alpha, w_0] = c\alpha(h_\alpha)w_0 = 2cw_0$.

so $n = 2c$.

Case 1: n is even, so h_α acts on W_0 by

$$\begin{bmatrix} n & & & & \\ & n-2 & & & \\ & & \ddots & & \\ & & & n-2 & \\ & & & & n \end{bmatrix}$$

so 0 is a weight, which is a contradiction since $W_0 \subseteq \sum_{c \in \mathbb{C}} \mathfrak{g}_{c\alpha}$

Case 2: n is odd so h_α acts by

$$\begin{bmatrix} n & & & & \\ & n-2 & & & \\ & & \ddots & & \\ & & & n-2 & \\ & & & & n \end{bmatrix}$$

and 1 is a weight. If $[h_\alpha, v] = v$ then $v \in \mathfrak{g}_{\frac{\alpha}{2}}$ so $\beta := \frac{\alpha}{2} \in \Phi$. Consider the s_β action on $V = \mathfrak{t} \oplus \sum_{c \in \mathbb{C}} \mathfrak{c}\alpha$. As an s_β representation, $V = \mathrm{Ker}(\beta) \oplus s_\beta \oplus W'$, with W' an s_β subrep. Note that $\mathfrak{g}_\alpha \subseteq W'$. If $\in \mathfrak{g}_\alpha$,

$$[h_\beta, x] = \alpha(h_\beta)x = 2\beta(h_\beta)x = 4x$$

so h_β acts on $\mathfrak{g}_\alpha$ with an even weight. So zero is a h_β-weight of W', which is a contradiction, since h_β acts by non-zero scalars.

□ (Claim 3, Theorem 9.7)

Putting this together, we have that $\dim(\mathfrak{g}_\alpha) = \dim(\mathfrak{g}_{-\alpha}) = 1$. □

Corollary 9.9. *: If $\alpha, c\alpha \in \Phi$ for some constant c then $c = \pm 1$.*

Theorem 9.10. *Suppose $\alpha, \beta \in \Phi$*

1. *$\beta(h_\alpha) \in \mathbb{Z}$.*

2. *The space $\bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\beta + k\alpha}$ is an irreducible representation of m_α.*

   In particular, the set $\{\beta + k\alpha : k \in \mathbb{Z}\} \cap \Phi$ is of the form $\beta - p\alpha, \beta - (p-1)\alpha, \cdots, \beta\beta + \alpha, \cdots, \beta + q\alpha$ for some $p, q \in \mathbb{Z}$. This is called the α-string through β.

3. *For p, q as in (2), $p - q = \beta(h_\alpha)$.*

4. *$[\mathfrak{g}_\alpha \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.*

Proof. 1. Let $V = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\beta + k\alpha}$, and let m_α act on V by restriction of the adjoint representation. Let

$$q = \max\{k \mid \beta + k\alpha \in \Phi\},$$

and let $v \in \mathfrak{g}_{\beta+k\alpha}$ be nonzero. Then

$$[e_\alpha, v] \in \mathfrak{g}_{\beta+(q+1)\alpha} = 0, \ [h_\alpha, v] \in \langle v \rangle,$$

and so v is a highest weight vector of weight $(\beta + q\alpha)(h_\alpha)$. So by the representation theory of $\mathfrak{sl}_2$,

$$\beta(h_\alpha) + q\alpha(h_\alpha) \in \mathbb{Z}_{\geq 0}$$

so $\beta(h_\alpha) + 2q \in \mathbb{Z}_{\geq 0}$ so $\beta(h_\alpha) \in \mathbb{Z}$.

2. $W := \{v, e_{-\alpha}v, e_{-\alpha}^2 v, \ldots\}$ is an irreducible subrepresentation of V, and h_α acts on W by

$$\begin{bmatrix} (\beta + q\alpha)(h_\alpha) & & & \\ & (\beta + (q-1)\alpha)(h_\alpha) & & \\ & & \ddots & \\ & & & -(\beta + q\alpha)(h_\alpha) \end{bmatrix}.$$

In particular,

$$W = \sum_{k=-p}^{q} \mathfrak{g}_{\beta+k\alpha}$$

for some $p \in \mathbb{Z}_{\geq 0}$. Suppose $W' \subseteq V$ is a subrepresentation not equal to W. Then W' contains a highest weight vector $w \in \mathfrak{g}_\gamma$ for some γ. Then

$$\gamma(h_\alpha) < -(\beta + q\alpha)(h_\alpha) \leq 0,$$

which contradicts (1).

3. $-(\beta + q\alpha)(h_\alpha) = (\beta - p\alpha)(h_\alpha)$.

4. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ so if $\mathfrak{g}_{\alpha+\beta} = 0$ we are done. If $\alpha+\beta \in \Phi$, take any nonzero $v \in \mathfrak{g}_\beta$. Then if $[e_\alpha, v] = 0$, v is a highest weight vector for V, a contradiction. So $[e_\alpha, v]$ is nonzero and therefore spans $\mathfrak{g}_{\alpha+\beta}$.

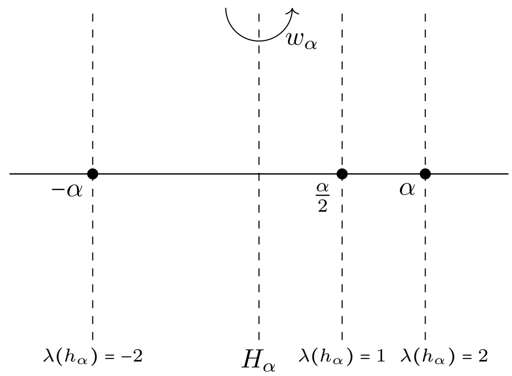
□

Corollary 9.11. *For $\alpha \in \Phi$, define $w_\alpha : \mathfrak{t}^* \longrightarrow \mathfrak{t}^*$ by $w_\alpha(\lambda) = \lambda - \lambda(h_\alpha)\alpha$. Then $w_\alpha(\Phi) = \Phi$.*

Proof. Let $\beta \in \Phi$. Let p, q be as in Theorem 9.10. We need to show $\beta - \beta(h_\alpha)\alpha \in \Phi$, and have

$$\beta - \beta(h_\alpha)\alpha = \beta - (p - q)\alpha$$

Since $-p \leq -(p-q) \leq q$, this is in the root string. □



Stepping back, w_α is reflection over the hyperplane $H_\alpha := \{\lambda \in \mathfrak{t}^* \mid \lambda(h_\alpha) = 0\}$, and this reflection preserves Φ. Our goal now will be to define a root system as something having all the nice properties of Φ and then show that root systems correspond to semisimple Lie algebras.

Root Systems

Roots in Euclidean Space

Proposition 10.1. *Define a bilinear form on* $\mathfrak{t}^*$ *by* $(\lambda, \mu) = K(t_\lambda, t_\mu)$ *for each* $\lambda, \mu \in \mathfrak{t}^*$.

1. *If* $\alpha, \beta \in \Phi$, $(\alpha, \beta) \in \mathbb{Q}$.
2. *If* $\alpha_1, \cdots, \alpha_l \in \Phi$ *form a basis of* $\mathfrak{t}^*$ *then* Φ *is contained in* $\text{span}_{\mathbb{Q}}\{\alpha_1, \cdots, \alpha_l\}$.
3. *This bilinear form s positive definite on* $\mathbb{Q}\Phi = \text{span}_{\mathbb{Q}}\{\alpha \mid \alpha \in \Phi\}$.

Proof. c.f. Grojnowski Proposition 5.7. Note for (1): $\beta(h_\alpha) = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$. □

Let $E = \text{span}_{\mathbb{R}}(\Phi)$. Then E is a Euclidean vector space.

Abstract Root Systems

Let $(E, (\cdot,\cdot))$ be a Euclidean space. Given $\alpha \in E$, define $\check{\alpha} : E \longrightarrow \mathbb{R}$ by

$$\check{\alpha}(\lambda) = \frac{2(\alpha, \lambda)}{(\alpha, \alpha)}.$$

Definition. A finite subset $\Phi \subset E$ is a *root system* if

1. $0 \notin \Phi$, and Φ spans E.
2. If $\alpha, \beta \in \Phi$ then $\check{\beta}(\alpha) \in \mathbb{Z}$.

3. Define $w_\alpha : E \longrightarrow E$ by

$$w_\alpha(\lambda) = \lambda - \check{\alpha}(\lambda)\alpha$$

Then if $\alpha \in \Phi$, $w_\alpha(\Phi) = \Phi$.

4. If $\alpha, c\alpha \in \Phi$ for c a constant, then $c = \pm 1$.

Removing (4) gives a "non-reduced" root system, but we will not discuss these.

Notation. If $\mu \in E, \lambda \in E^*$ then $\langle \mu, \lambda \rangle = \lambda(\mu)$, so for example $\langle \alpha, \check{\beta} \rangle = \check{\beta}(\alpha)$.

Example. If $\mathfrak{g}$ is a semisimple Lie algebra, $\mathfrak{t} \subseteq \mathfrak{g}$ a Cartan subalgebra Φ the set of roots associated to $\mathfrak{t}$ then Φ forms a root system in $\mathbb{R}\Phi$.

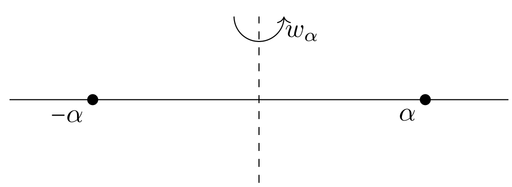
Definition. The *rank* of a root system is $\dim(E)$.

Definition. If $(\Phi, E), (\Phi, E')$ are root systems then an *isomorphism* is a linear isomorphism of vector spaces $\rho : E \longrightarrow E'$ with $\rho(\Phi) = \Phi'$ and $\langle \rho(\alpha), \rho(\check{\beta}) \rangle = \langle , \alpha, \check{\beta} \rangle$ for all $\alpha, \beta \in \Phi$.

Definition. If $\alpha \in \Phi$ then $\check{\alpha}$ is called a *coroot*.

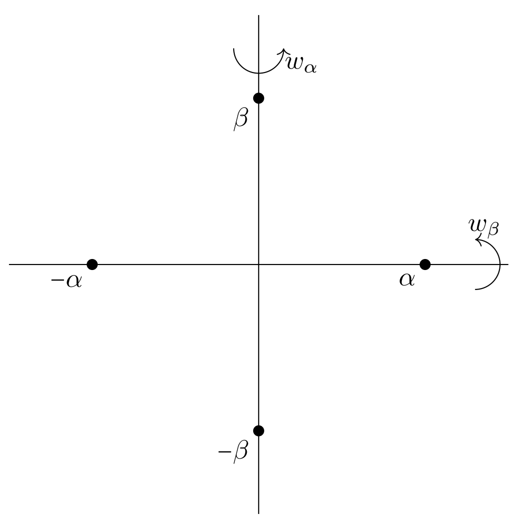
Examples. Rank 1

A_1: $\Phi = \{\alpha, -\alpha\}, \langle \alpha, \check{\alpha} \rangle = -2$.

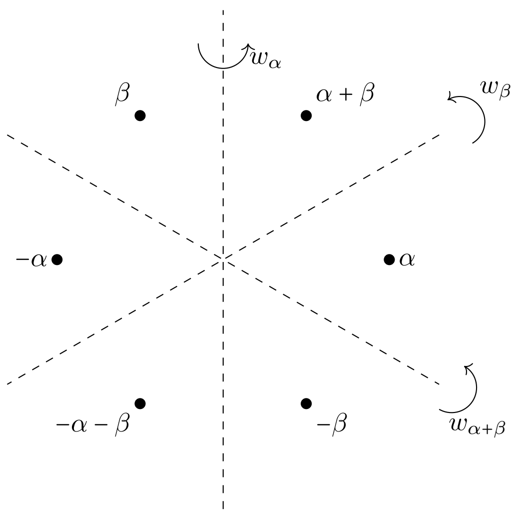


Rank 2

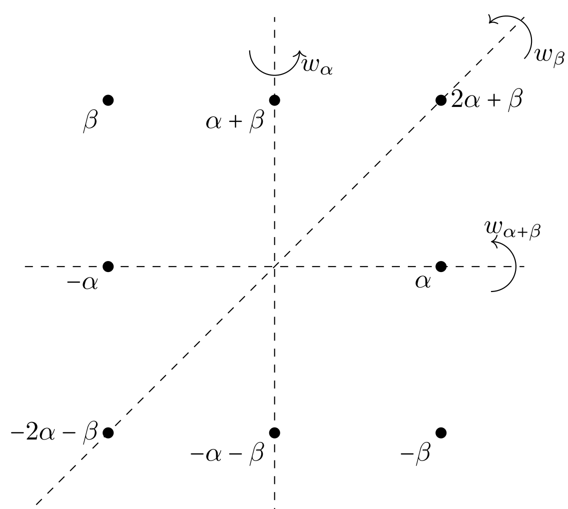
$A_1 \times A_1$: $\Phi = \{\alpha, -\alpha, \beta, -\beta\}, \langle \alpha, \check{\alpha} \rangle = \langle \beta, \check{\beta} \rangle = -2, \langle \alpha, \check{\beta} \rangle = 0$.



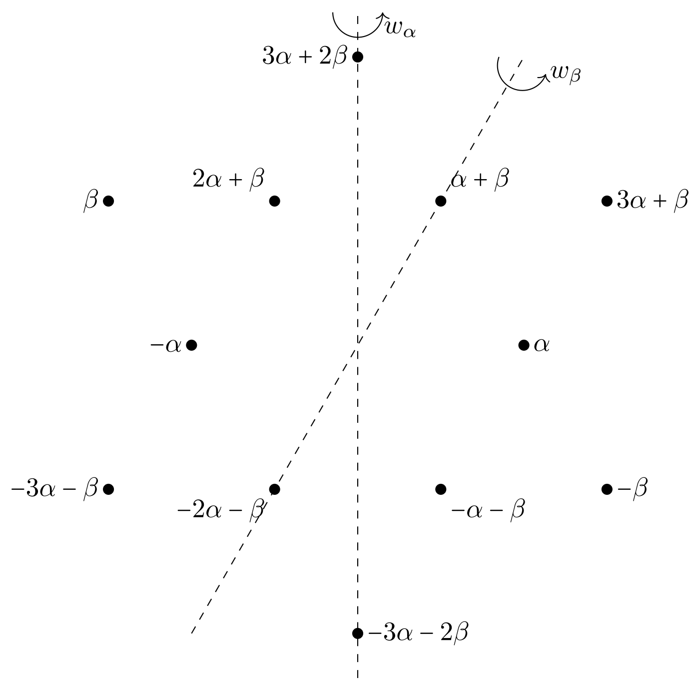
A_2: $\Phi = \{\alpha, -\alpha, \beta, -\beta, \alpha+\beta, -\alpha-\beta\}, (\alpha,\alpha) = (\beta,\beta) = 1, (\alpha,\beta) = -\frac{1}{2}$, so $\langle \alpha, \check{\beta} \rangle = \langle \beta, \check{\alpha} \rangle = -1$.



B_2: $\Phi = \{\alpha, -\alpha, \beta, -\beta, \alpha+\beta, 2\alpha+\beta, -\alpha-\beta, -2\alpha-\beta\}$. $\langle \alpha, \check{\beta} \rangle = -1$. $\langle \beta, \check{\alpha} \rangle = -2$.



G_2: $\langle \alpha, \check{\beta} \rangle = -1$, $\langle \beta, \check{\alpha} \rangle = -3$, $(\beta,\beta) = 3, (\alpha,\alpha) = 1$.



Returning to Lie algebras,

1. $\mathfrak{sl}_2 = \langle h \rangle \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ with $\alpha(h) = 2$. Taking h as the generator of the Cartan subalgebra, have root system corresponding to A_1.

2. $\mathfrak{sl}_3$, with

$$\mathfrak{t} = \left\langle \left\{ \begin{bmatrix} 1 & & \\ & -1 & \\ & & \end{bmatrix}, \begin{bmatrix} & & \\ & 1 & \\ & & -1 \end{bmatrix} \right\} \right\rangle$$

   has $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$ where $\alpha(h_1) = 2, \alpha(h_2) = -1, \beta(h_1) = -1, \beta(h_2) = 2$, then $\Phi \cong A_2$.

3. For $\mathfrak{sp}_4$ or $\mathfrak{so}_5$, $\Phi \cong B_2$.

4. G_2 corresponds to a Lie algebra that we are yet to define.

Definition. The *Weyl group* of a root system (Φ, E) is the subgroup of $\mathrm{GL}(E)$ generated by $\{w_\alpha \mid \alpha \in \Phi\}$.

Examples. 1. For A_1, $W \cong C_2$.

2. For B_2, $W \cong D_8$.

3. For A_2, $W \cong D_6$.

4. For G_2,, $W \cong D_{12}$.

Lemma 11.1. *The Weyl group of Φ is isomorphic to a subgroup of S_n where $n = |\Phi|$.*

Proof. W acts on Φ, and Φ spans E. □

Note that if $(\Phi_1, E_1), (\Phi_2, E_2)$ are root systems then $(\Phi_1 \cup \Phi_2, E_1 \oplus E_2)$ is also a root system.

Definition. A root system of this form with both Φ_i nonempty is called *reducible*, and otherwise Φ is *irreducible*.

Examples

1. $A_1 \times A_1$ is reducible.
2. A_1, A_2, B_2, G_2 are all irreducible.
3. If Φ corresponds to a Cartan subalgebra in a semisimple Lie algebra $\mathfrak{g}$ then Φ is irreducible exactly when $\mathfrak{g}$ is.

   Proof. (Exercise) Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be reducible, and let $\mathfrak{t}_1, \mathfrak{t}_2$ be Cartan subalgebras of components. Then $\mathfrak{t}_1 \oplus \mathfrak{t}_2$ is a Cartan subalgebra of $\mathfrak{g}$. Let Φ_1, Φ_2 be the roots of $\mathfrak{g}_1, \mathfrak{g}_2$ with respect to $\mathfrak{t}_1, \mathfrak{t}_2$. For $\alpha \in \Phi_i$, extend α to $\mathfrak{t}_1 \oplus \mathfrak{t}_2$ via $\bar{\alpha}(t_1 + t_2) = \alpha(t_i)$. Then $\bar{\alpha}$ is a root of $\mathfrak{g}$ with $\mathfrak{g}_{\bar{\alpha}} = \mathfrak{g}_{i_\alpha} \oplus 0$ so

   $$\mathfrak{g} = (\mathfrak{t}_1 \oplus \mathfrak{t}_2) \oplus \bigoplus_{\alpha \in \Phi_1 \cup \Phi_2} \mathfrak{g}_{\bar{\alpha}}$$

   so the root system of $\mathfrak{g}$ is $(\Phi_1 \cup \Phi_2, \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2)$ which is reducible.

   Now let $\Phi = \Phi_1 \cup \Phi_2$ be reducible corresponding to $\mathfrak{g} = E_1 \oplus E_2$. Since E_1, E_2 are vector spaces it suffices to check that they are closed under the bracket as subspaces of $\mathfrak{g}$, and that $[e_1, e_2] = 0$ for any $e_i \in E_i$. Any e_2 is in the span of the root spaces corresponding to Φ_2. □

Lemma 11.2. *If Φ is a root system and $\alpha, \beta \in \Phi$, $\alpha \neq \pm\beta$ then*

$$\langle \alpha, \check{\beta} \rangle \langle \beta, \check{\alpha} \rangle \in \{0, 1, 2, 3\}.$$

Proof. Recall $(\alpha, \beta) = \sqrt{(\alpha, \alpha)}\sqrt{(\beta, \beta)} \cos\theta$ where θ is the angle between α and β. So

$$\langle \alpha, \check{\beta} \rangle \langle \beta, \check{\alpha} \rangle = \frac{4(\alpha, \beta)^2}{(\alpha, \alpha)(\beta, \beta)} = 4\cos^2\theta \in \mathbb{Z}.$$

Now, $\cos^2\theta \in [0, 1]$ so $\cos^2\theta \in \{0, \frac{1}{4}, \frac{1}{3}, \frac{3}{4}, 1\}$, but since $\alpha \neq \pm\beta$, $\cos^2\theta \neq 1$. □

Corollary 11.3. *If Φ is a root system and α, β are roots then $\langle \alpha, \check{\beta} \rangle \in \{0, \pm 1, \pm 2, \pm 3\}$.*

Proof. Immediate from the above. □

Exercise. The only rank two systems are, up to isomorphism, those listed above.

Proof. It is clear that A_1 is the only rank 1 root system. Since any reducible root system of rank two must be a direct sum of two rank 1 systems, $A_1 \times A_1$ is the only reducible rank two system.

For the irreducible rank 2 systems, we must have two basis elements α, β, and we may choose α to minimise (α, α), and β to minimise $\langle \beta, \check{\alpha} \rangle$ subject to $\beta \neq -\alpha$. Since α is of minimal length, $\langle \alpha, \check{\beta} \rangle \leq \langle \beta, \check{\alpha} \rangle$ and since α, β are linearly independent these two brackets have product in $\{0,1,2,3\}$ by Lemma 11.2. Also, $\beta \in \Phi$ so $-\beta \in \Phi$, we have $\langle \alpha, \check{\beta} \rangle \leq 0$, and so $\langle \beta, \check{\alpha} \rangle \leq 0$. The possibilities are then:

$\langle \alpha, \check{\beta} \rangle$	$\langle \beta, \check{\alpha} \rangle$	Root System
0	0	$A_1 \times A_1$
−1	−1	A_2
−1	−2	B_2
−1	−3	G_2

And the root systems *are* determined up to isomorphism by the pairing of α and β, since these determine the position of β relative to α, which then determines w_β, and the action of w_α, w_β on $\pm\alpha, \pm\beta$ generates all of each of the root systems listed. Moreover, subject to the assumptions on α, β, no other roots can be added. ☐

Corollary 11.4. *If Φ is an irreducible root system then (α, α) can take at most two values as α varies over Φ.*

Proof. (Sketch) If we can have three values, then there are $\alpha, \beta \in \Phi$ with

$$\frac{(\alpha, \alpha)}{(\beta, \beta)} = \frac{2}{3}$$

so

$$\frac{\langle \alpha, \check{\beta} \rangle}{\langle \beta, \check{\alpha} \rangle} = \frac{2}{3}$$

so

$$6 \mid \langle \alpha, \check{\beta} \rangle \langle \beta, \check{\alpha} \rangle$$

which is a contradiction. Care is required to justify why the product can be taken to be nonzero. ☐

Definition. An irreducible Φ is called simply laced if (α, α) takes exactly one value as α varies over Φ.

Exercise. Φ is simply laced if and only if $\langle \alpha, \check{\beta} \rangle \in \{0, \pm 1\}$ for all α, β with $\alpha \neq \pm\beta$.

Proof. See Example Sheet. ☐

Weyl Chambers, Root Bases

Throughout, (Φ, E) is a root system, and for $\alpha \in \Phi$, $H_\alpha = \{\lambda \in E \mid \langle \lambda, \check{\alpha} = 0\}$ is the corresponding root hyperplane.

Definition. The connected components of $E - \bigcup_{\alpha \in \Phi} H_\alpha$ are called *Weyl chambers.*

Definition. A subset $\Delta = \{\alpha_1, \cdots, \alpha_l\} \subseteq \Phi$ is called a *root basis* if:

1. Δ forms a basis for E;
2. $\alpha = \sum_{i=1}^{l} c_i \alpha_i \in \Phi$ so $c_i \in \mathbb{Z}_{\geq 0} \forall i$ or $c_i \in \mathbb{Z}_{\leq 0} \forall i$.

Example. In our rank 2 systems, $\{\alpha, \beta\}$ form a root basis.

Definition. If $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ is a root basis, the α_i are called *simple roots.* If $\alpha = \sum_{i=1}^{l} c_i \alpha_i$ with $c_i \geq 0$, α is a *positive root*, and if $c_i \leq 0$ α is a *negative root.* Φ^+ is the set of all positive roots, Φ^- the set of all negative roots.

Warm Up for Lecture 13: Let W be the Weyl group of (Φ, E). Then if Δ is a root basis, $w \in W$, then $w(\Delta)$ is a root basis.

Proof. We know $w \in \mathrm{GL}(E)$ so $w(\Delta)$ is a basis for E and a subset of Φ. If $\alpha \in \Phi$, $\alpha = \sum c_i \alpha_i$ for $c_i \geq 0$ for all i or $c_i \leq 0$ for all i, so $w(\alpha) = \sum c_i w(\alpha_i)$ with $c_i \geq 0$ for all i or $c_i \leq 0$ for all i. □

The warm up tells us that W acts on the set of root bases. The goal for now will be to construct a root basis. Our set up will be as follows.

Choose $\gamma \in E - \bigcup_{\alpha \in \Phi} H_\alpha$ and define

$$\Phi_\gamma^+ = \{\alpha \in \Phi \mid \langle \gamma, \check{\alpha} \rangle > 0\} = \{\alpha \in \Phi \mid (\gamma, \alpha) > 0)\}.$$

Set $\Phi_\gamma^- = -\Phi_\gamma^+ (= \{\alpha \in \Phi \mid \langle \gamma, \check{\alpha} \rangle\}$. Define also

$$\Delta_\gamma = \{\alpha \in \Phi_\gamma^+ \mid \ \alpha \neq \beta_1 + \beta_2 \text{ for any } \beta_1, \beta_2 \in \Phi_\gamma^+\}.$$

Note that these sets only depend on the Weyl chamber of γ.

Theorem 12.1. *1. Δ_γ is a root basis.*

2. Every root basis is of the form Δ_γ for some $\gamma \in E - \bigcup_{\alpha \in \Phi} H_\alpha$.

Proof of (1): We split into the following claims.

Claim 1: If $\alpha, \beta \in \Delta_\gamma$, $\alpha - \beta \notin \Delta_\gamma$.

Suppose $\alpha, \beta \in \Phi$. Without loss of generality $\alpha - \beta \in \Phi_\gamma^+$, else take $\beta - \alpha$. Then $\alpha = (\alpha - \beta) + \beta$, contradicting the definition of Δ_γ.

Claim 2: If $\alpha, \beta \in \Delta_\gamma$ and $\alpha \neq \beta$ then $\langle \alpha, \check{\beta} \rangle = 0$.

Recall $\langle \alpha, \check{\beta} \rangle \langle \beta, \check{\alpha} \rangle \in \{0, 1, 2, 3\}$. Suppose $\langle \alpha, \check{\beta} \rangle > 0$. Without loss of generality it is 1, else consider $\langle \beta, \check{\alpha} \rangle$. $w_\beta(\alpha) = \alpha - \langle \alpha, \check{\beta} \rangle \beta = \alpha - \beta \in \Phi$ since W preserves Φ, which contradicts claim 1.

Claim 3: Let $\Delta_\gamma = \{\alpha_1, \cdots, \alpha_l\}$. If $\alpha \in \Phi_\gamma^+$, $\alpha = \sum_{i=1}^l c_i \alpha_i$ then $c_i \in \mathbb{Z}_{\geq 0}$.

Suppose there is some α that cannot be written this way. Pick such an α with (γ, α) minimal. $\alpha \notin \Delta_\gamma$ so $\alpha = \beta_1 + \beta_2$ for $\beta_1, \beta_2 \in \Phi_\gamma^+$. Now,

$$(\alpha, \gamma) = (\beta_1 + \gamma) + (\beta_2, \gamma),$$

so

$$(\beta_i, \gamma) < (\alpha, \gamma),$$

so β_1, β_2 can be written as a $\mathbb{Z}_{\geq 0}$-linear combination of the α_i, so as the sum of β_1, β_2, so too can α, a contradiction.

Note that this implies that every element in Φ_γ^- is a $\mathbb{Z}_{\leq 0}$ combination of α_i and that Δ_γ spans E, since Φ does.

Claim 4: Δ_γ forms a linearly independent set.

Suppose that for some $c_i \in \mathbb{R}$, $\sum_{i=1}^l c_i \alpha_i = 0$. Without loss of generality $c_i \geq 0$ for $1 \leq i \leq m$ and $c_i \leq 0$ for $m + 1 \leq i \leq l$, so

$$v := \sum_{i=1}^m c_i \alpha_i = -\sum_{j=m+1}^l c_j \alpha_j,$$

so

$$0 \leq (v, v) = -\sum_{i,j} c_i c_j (\alpha_i, \alpha_j) \leq 0,$$

since $c_i c_j \leq 0$ and by claim 2, $(\alpha_i, \alpha_j) \leq 0$, so $v = 0$. So

$$0 = (\gamma, v) = \sum_{i=1}^m c_i (\gamma, \alpha_i),$$

so $c_i = 0$ for $1 \leq i \leq m$ and similarly for $m + 1 \leq j \leq l$.

For (2), see Humphreys

□

Corollary 12.2. *There is a bijection*

$$\{\textit{Weyl chambers}\} \leftrightarrow \{\textit{root bases}\}.$$

Proof. Given a Weyl chamber C, choose $\gamma \in C$ and choose root basis Δ_γ. Given $\Delta = \Delta_\gamma$, $\gamma \in C$ for some Weyl chamber C. □

Definition. Given a root basis Δ_γ, the *fundamental Weyl chamber* is the Weyl chamber containing γ.

Definition. If $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ is a root basis and $\alpha \in \Phi$ then the *height* of α is $\sum c_i$ where $\alpha = \sum_{i=1}^{l} c_i \alpha_i$.

Proposition 12.3. *If $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ is a root basis and $\beta \in \Phi^+ \backslash \Delta$, there is an i such that $\beta - \alpha_i \in \Phi$.*

Proof. Given β, if $(\beta, \alpha_i) \leq 0$ for all i then $\Delta \cup \{\beta\}$ is an linearly independent set by the proof of claim 4 so there is an i such that $\langle \beta, \check{\alpha_i} \rangle > 0$. Since $\langle \beta, \check{\alpha_i} \rangle \langle \alpha_i, \check{\beta} \rangle \in \{0, 1, 2, 3\}$ then $\langle \beta, \check{\alpha_i} \rangle = 1$ or $\langle \alpha_i, \check{\beta} \rangle = 1$.

So, $w_{\alpha_i}(\beta) = \beta - \alpha_i \in \Phi$ or $w_\beta(\alpha_i) = \alpha_i - \beta \in \Phi$ so $\beta - \alpha_i \in \Phi$. □

Corollary 12.4. *If $\beta \in \Phi^+$, then β can be written as $\beta = \alpha_{i1} + \cdots + \alpha in$ with $\alpha_i j$ a simple root for each j, and $\sum_{j=1}^{k} \alpha_{ij}$ a root for all k.*

Proof. By induction on height, using Proposition 12.3. □

Note: Corollary 12.4 implies that for $\mathfrak{g}$ a semisimple Lie algebra with Cartan subalgebra $\mathfrak{t}$, root system Φ, then given a root basis Φ, $\mathfrak{g}$ is generated by $\{e_\alpha, e_{-\alpha} \mid \alpha \in \Phi\}$.

From the warm up, we know that W acts on the set of root bases, and therefore preserves the set of Weyl chambers.

Proposition 12.5. *If Δ is a root basis and $w \in W$ $C_{w(\Delta)} = w(C_\Delta)$.*

Proof. Uses

Lemma 12.6. *If $w \in W$ and $\lambda, \mu \in E$ then $\langle \lambda, \check{\mu} \rangle = \langle w(\lambda), w(\check{\mu}) \rangle$.*

□

Warm Up for Lecture 14: For Φ a root system, Δ a root basis and W Weyl group, $\alpha \in \Delta$, w_α permutes $\Phi^+ - \Delta$.

Proof. Suppose $\alpha = \alpha_1$ and $\Delta = \{\alpha_1, \cdots, \alpha_l\}$. Suppose $\beta \in \Phi^+ - \{\alpha\}$. Then $\beta = \sum c_i \alpha_i$ with c_i non-negative integers.

$$w_{\alpha_1}(\beta) = \beta - \langle \beta, \check{\alpha_1} \rangle \alpha_1 = (c_1 - \langle \beta, \check{\alpha_1} \rangle)\alpha_1 + \sum_{i=2}^{l} c_i \alpha_i.$$

Since β is a positive root, and $\beta \neq \alpha_1$ so $w_{\alpha_1}(\beta) \neq -\alpha_1$, $c_i > 0$ for some $i > 1$ and so $w_{\alpha_1}(\beta)$ is a positive root different from α, since $\beta \neq -\alpha$, which is what we wanted to show. □

Theorem 12.7. *1. The Weyl group acts simply transitively on the set of root bases (and the set of Weyl chambers).*

2. Given a root basis Δ and $\alpha \in \Phi$ there is a $w \in W$ with $w(\alpha) \in \Delta$. This is not necessarily unique.

3. If $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ is a root basis then W is generated by $\{w_{\alpha_i} \mid 1 \leq i \leq l\}$.

Proof. See Humphrey's 10.3. □

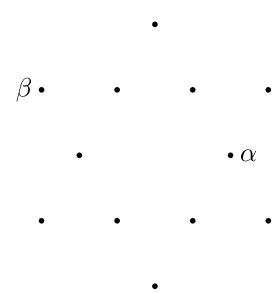
8 Classification of Irreducible Root Systems

Throughout, (Φ, E) is a root system, $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ is a root basis and W is the Weyl group of Φ.

Definition. The *Cartan matrix* of Φ is the $l \times l$ matrix $(a_{i,j})$ with $a_{i,j} = \langle \alpha_i, \check{\alpha_j} \rangle$.

Note that this is independent of choice of root basis since given Δ' there is a $w \in W$ with $w(\Delta) = \Delta'$ preserving angle brackets.

Example. For G_2,



$\alpha_1 = \alpha, \alpha_2 = \beta$ we have $\langle \alpha_1, \check{\alpha_2} \rangle = -1, \langle \alpha_2, \check{\alpha_1} \rangle = -3$ and so we have Cartan matrix $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ with uniqueness up to reordering the base.

Proposition 12.8. *Suppose* (Φ', E') *is a root basis* $\{\alpha_1', \cdots, \alpha_l'\}$ *such that*

$$\langle \alpha_i, \check{\alpha_j} \rangle = \langle \alpha_i', \check{\alpha_j'} \rangle \text{ for all } i, j.$$

Then the linear map defined by $\alpha_i \longrightarrow \alpha_i'$ *induces an isomorphism of root systems.*

Proof. $\alpha_i \longrightarrow \alpha_i'$ induces an isomorphism of vector spaces $\psi : E \longrightarrow E'$. We need to show:

1. $\psi(\Phi) = \Phi$.
2. $\langle \psi(\alpha_i), \check{\psi(\beta)} \rangle = \langle \alpha, \check{\beta} \rangle$ for all $\alpha, \beta \in \Phi$.

Consider the action of W.

$$\begin{aligned} \psi(w_{\alpha_i}(\alpha_j)) &= \psi(\alpha_j - \langle \alpha_j, \check{\alpha_i} \rangle \alpha_i) \\ &= \alpha_j' - \langle \alpha_j, \check{\alpha_i} \rangle \alpha_i' \\ &= w_{\alpha_i'}(\alpha_j') \\ &= w_{\alpha_i'}(\psi(\alpha_j)). \end{aligned}$$

So, the Weyl group of Φ, Φ' are isomorphic since both are generated by simple reflections, and $\psi(w(\alpha)) = w(\psi(\alpha))$ for each $\alpha \in \Phi$ under this identification.

1. Given $\alpha \in \Phi$ there is $w \in W$ with $w(\alpha) \in \Delta$. $\psi(w(\alpha)) \in \Delta'$ so $w(\psi(\alpha)) \in \Delta'$ so $\psi(\alpha) \in \Phi'$. For the other containment do the same with ψ^{-1}.
2. Given $\alpha, \beta \in \Phi$ choose $w \in W$ with $w(\beta) \in \Delta$. Write $w(\alpha) = \sum c_i \alpha_i'$. Then

$$\begin{aligned}\langle \alpha, \check{\beta} \rangle &= \langle w(\alpha), w(\check{\beta}) \rangle \\ &= \sum c_i \langle \alpha_i, \check{\beta} \rangle \\ &= \sum c_i \langle \psi(\alpha_i), \psi(w(\check{\beta})) \rangle \\ &= \langle w(\psi(\alpha)), w(\psi(\check{\beta})) \rangle.\end{aligned}$$

□

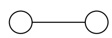
Definition. The *Dynkin diagram* of Φ has:

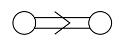
1. vertices $\leftrightarrow \Delta$
2. The ith and jth vertices connected by $\langle \alpha_i, \check{\alpha_j} \rangle \langle \alpha_j, \check{\alpha_i} \rangle$ edges.
3. If a multiple edge occurs, an arrow points to the shorter root.

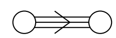
Examples. For ranks 1 and 2 we have the following.

Type A_1: 

Type $A_1 \times A_1$: 

Type A_2: 

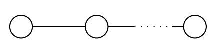
Type B_2: 

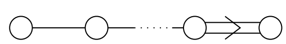
Type G_2: 

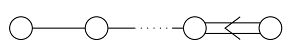
Note that the maximal number of edges between any pair of vertices is 3, and Φ is simply laced if and only if the Dynkin diagram has no multiple edges.

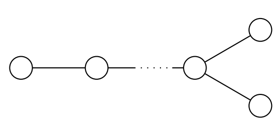
Exercise. Φ is irreducible if and only if the Dynkin diagram is simply connected.

Theorem 12.9. *If Φ is irreducible, then its Dynkin diagram is one of either a diagram associated to the classical root systems:*

A_l: 

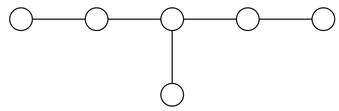
B_l: 

C_l: 

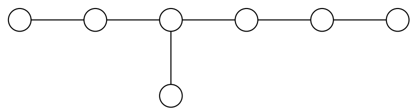
D_l: 

or it is one of the exceptional root systems:

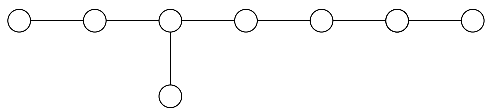
E_6:

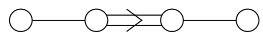


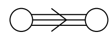
E_7:



E_8:



F_4: 

G_2: 

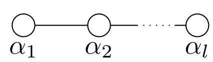
Proof. See Humphreys 11.4. □

Theorem 12.10. *For every Dynkin diagram $\mathcal{D}$ listed, there is a simple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{t}$, roots Φ corresponding to $\mathfrak{t}$ such that the Dynkin diagram of Φ is given by $\mathcal{D}$.*

Proof. (Sketch) For A_l, let $\mathbf{e}_i$ be the ith standard basis vector for $\mathbb{R}^{l+1}$. Let $\Phi = \{\mathbf{e}_i - \mathbf{e}_j \mid i \neq j\} \subseteq \mathbb{R}^{l+1}$. Φ spans an r-dimensional subspace of $\mathbb{R}^{l+1}$, call it E. Then Φ is a root system in E with root basis given by $\{\mathbf{e}_i - \mathbf{e}_{i+1} \mid 1 \leq i \leq l\}$ and

$$\langle \alpha_i, \check{\alpha_j} \rangle = \begin{cases} -1 & i, j \text{ differ by } 1 \\ 2 & i = j \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding Dynkin diagram is



so Φ is type A_l. Now, w_{α_i} flips the ith and $(i+1)$th co-ordinate, so $W \cong S_{l+1}$. The corresponding Lie algebra is $\mathfrak{sl}_{l+1}$ with Cartan subalgebra $\begin{bmatrix} * & & \\ & \ddots & \\ & & * \end{bmatrix}$ and

$$\alpha_i\left(\begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_{l+1} \end{bmatrix}\right) = t_i - t_{i+1}.$$

For the classical root systems, for $\mathbf{e}_i$ basis of $\mathbb{R}^l$ and $\mathfrak{t}$ diagonal subalgebra we get

Type	$\Phi \subseteq \mathbb{R}^l$	$\Delta \in \Phi$	W	$\mathfrak{g}$
B_l	$\{\pm\mathbf{e}_i, \pm\mathbf{e}_i \pm \mathbf{e}_j, i \neq j\}$	$\{\mathbf{e}_i - \mathbf{e}_{i+1} \mid 1 \leq i \leq l\} \cup \{\mathbf{e}_l\}$	$S_l \ltimes C_2^l$	$\mathfrak{so}_{2l+1}$
C_l	$\{\pm 2\mathbf{e}_i, \pm\mathbf{e}_i \pm \mathbf{e}_j, i \neq j\}$	$\{\mathbf{e}_i - \mathbf{e}_{i+1} \mid 1 \leq i \leq l\} \cup \{2\mathbf{e}_l\}$	$S_l \ltimes C_2^l$	$\mathfrak{sp}_{2l}$
D_l	$\{\pm\mathbf{e}_i \pm \mathbf{e}_j, i \neq j\}$	$\{\mathbf{e}_i - \mathbf{e}_{i+1} \mid 1 \leq i \leq l\} \cup \{\mathbf{e}_{l-1} + \mathbf{e}_l\}$	$S_l \ltimes C_2^{l-1}$	$\mathfrak{so}_{2l}$

Weyl group for B_l: S_l acts on co-ordinates, and each C_2 acts as a sign change on each co-ordinate.

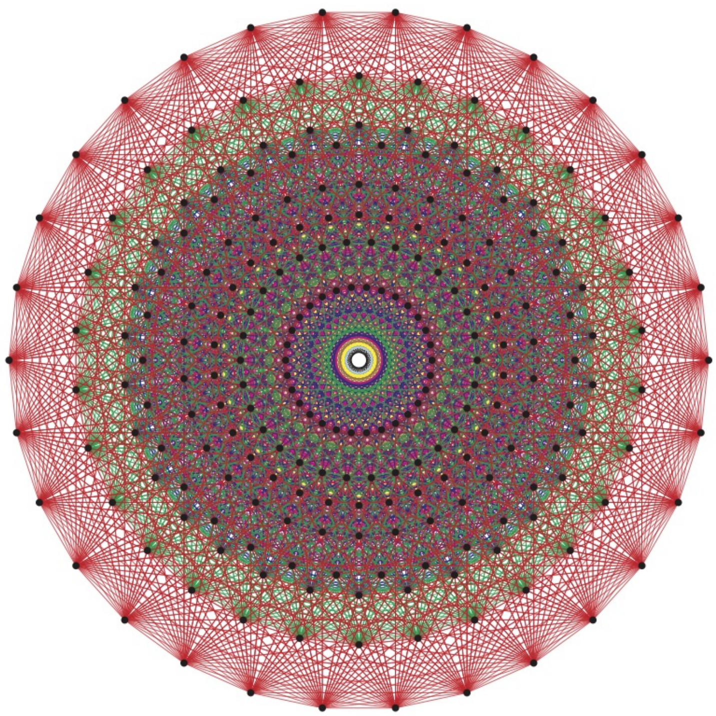
- G_2 - know root system.
- $F_4 \subseteq \mathbb{R}^4$, $\Phi = \{\pm\mathbf{e}_i, \pm\mathbf{e}_i \pm \mathbf{e}_j, \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4) i \neq j\}$.
- E_6, E_7, E_8, see Grojowski's notes section 6.

□

$E_8 \subseteq \mathbb{R}^8$, $|\Phi| = 240$.
Let

$$w_c = \prod_{i=1}^{8} w_{\alpha_i}$$

called a *coxeter element* of W. The order of w_c is 30 and there is a plane of $\mathbb{R}^8$ on which it acts by rotation. The picture below shows projection of roots to that plane. The circles are the orbits under the group generated by w_c.



In general, to look up root systems, use the `spherical explorer`.

For computations, it is best to know things in terms of simple roots.

The exceptional Lie algebras are written as $\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$. $\mathfrak{g}_2$ is the algebra of derivations of the octonians $\mathbb{O}$, where a *derivation* is a linear map $\delta, \delta(ab) = \delta(a)b + a\delta(b)$. $\mathbb{O}$ is 8-dimensional. It has a one-dimensional center on which

$\mathfrak{g}_2$ acts trivially. We can get a representation $\mathfrak{g}_2 \hookrightarrow \mathfrak{so}_7$, the lowest dimensional nontrivial representation (see Humphrey's 19.3). Others can be constructed, see Fulton Harris 22.4.

Note that given a root system Φ there is a natural construction of a Lie algebra with that root system.

To summarise, so far we have

$$\{\mathfrak{g} \text{ simple}, \mathfrak{t} \text{ CSA}\} \twoheadrightarrow \{\text{irreducible root systems } \Phi\} \leftrightarrow \{\text{connected Dynkin diagrams}\}.$$

Next,

1. We'll show the root system corresponding to $\mathfrak{g}$ is independent of choice of Cartan subalgebra.

2. We'll show two Lie algebras with the same root system are isomorphic.

Isomorphism and Conjugacy

Throughout, $\mathfrak{g}$ is a semisimple Lie algebra, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$, Φ root system corresponding to $\mathfrak{t}$ and $\Delta \subseteq \Phi$ a root basis.

Proposition 13.1. *If $\mathfrak{t}'$ is another Cartan subalgebra of $\mathfrak{g}$ then there is an inner automorphism $\psi \in G_{\text{ad}}$ with $\psi(\mathfrak{t}) = \mathfrak{t}'$.*

Proof. see Humphreys 16.4 □

Definition. The *rank* of $\mathfrak{g}$ is the dimension of a Cartan subalgebra, which is independent of choice of Cartan subalgebra by Proposition 14.1.

Corollary 13.2. *If $\mathfrak{t}'$ is a Cartan subalgebra of $\mathfrak{g}$ with root system Φ then Φ, Φ' are isomorphic.*

Proof. Take ψ as in Proposition 14.1. Suppose $t \in \mathfrak{t}$, $\alpha \in \Phi$, $e_\alpha \in \mathfrak{g}_\alpha$. Then

$$[\psi(t), \psi(e_\alpha)] = \psi([t, e_\alpha]) = \psi(\alpha(t)e_\alpha) = \alpha(t)(\psi(e_\alpha)).$$

Now, $\psi(e_\alpha)$ spans a root space for $\mathfrak{t}'$ so $\Phi' = \{\alpha \circ \psi^{-1} \mid \alpha \in \Phi\}$. □

Theorem 13.3. *If $\mathfrak{g}'$ is a semisimple Lie algebra with root system Φ then $\mathfrak{g} \cong \mathfrak{g}'$.*

Proof. Follows from the theory of finite structure constants (see Carter, Lie algebras of Finite and Affine Type, section 7).

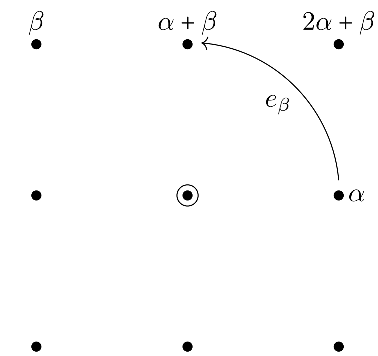
The idea is to choose a basis h_{α_i} of $\mathfrak{t}$ and choose e_α in each root space so that for each α, $[e_\alpha, e_{-\alpha}] = h_\alpha$. This gives a basis of the Lie algebra. Then:

$$[h_{\alpha_i}, h_{\alpha_j}] = 0 \quad [h_{\alpha_i}, e_\alpha] = \alpha(h_{\alpha_i}) \quad [e_\alpha, e_\beta] = \begin{cases} N_{\alpha\beta} e_{\alpha+\beta} & \alpha + \beta \in \Phi \\ h_\alpha & \beta = -\alpha \\ 0 & \alpha + \beta \notin \Phi \cup \{0\} \end{cases}$$

where $N_{\alpha\beta}$ are the structure constants. □

Warm Up for Lecture 16: Let $\mathfrak{g} = \mathfrak{so}_5$ α, β simple roots for the root system of $\mathfrak{g}, \Phi$. Recall that $m_\alpha = \mathfrak{g}_\alpha \oplus \langle [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \rangle \oplus \mathfrak{g}_{-\alpha} \cong \mathfrak{sl}_2$. Decompose the adjoint representation of $\mathfrak{g}$ under the action of m_α, m_β.

The following diagram has a dot for each root space, and the circle around 0 notates that $\mathfrak{t} = \mathfrak{g}_0$ his dimension 2.



Suppose $e_\alpha \in \mathfrak{g}_\alpha$. Then $e_\alpha \cdot \mathfrak{g}_\gamma = \mathfrak{g}_{\lambda+\gamma}$ for all $\gamma \in \Phi$, so each α-root string corresponds to an irreducible subrepresentation (m_α) of $\mathfrak{g}$.

$$\mathfrak{g}\,|_{m_\alpha} = V(2) \oplus V(2) \oplus V(2) \oplus V(0),$$

$$\mathfrak{g}\,|_{m_\beta} = V(2) \oplus V(1) \oplus V(1) \oplus V(0) \oplus V(0) \oplus V(0).$$

Weights

Let (Φ, E) be a root system and fix a root basis $\Delta = \{\alpha_1, \cdots, \alpha_l\}$.

Definition. The *root lattice* $\mathbb{Z}\Phi$ is $\{\sum_{\alpha \in \Phi} c_\alpha \alpha \mid \; c_\alpha \in \mathbb{Z}\} \subseteq E$. The *weight lattice* X is

$$X = \{\lambda \in E, \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi\},$$

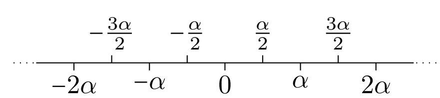
and the elements of the weight lattice are the *weights*.

Note that

1. $\mathbb{Z}\Phi \subseteq X$.

2. If $\lambda \in X$ so is $w(\lambda)$ for all $w \in W$, since $\langle \lambda, \check{\alpha} \rangle = \langle w(\lambda), w(\check{\alpha}) \rangle$.

   Example. For A_1, the root lattice $\mathbb{Z}\Phi$ is shown below the line, and $X \backslash \mathbb{Z}\Phi$ is shown above.



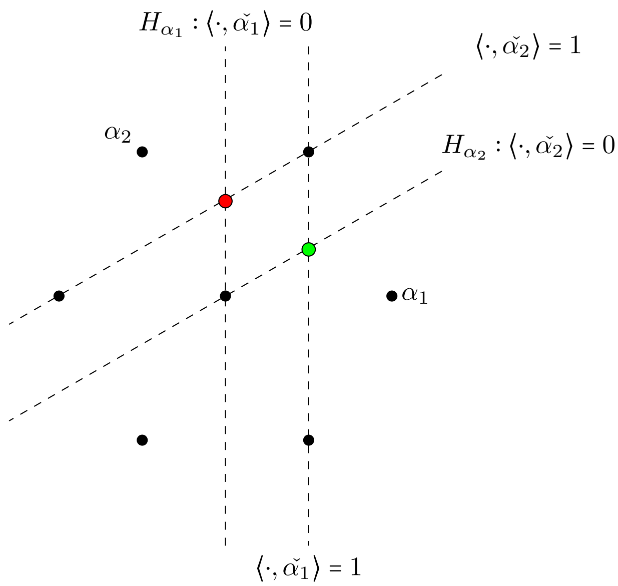
Lemma 14.1. $X = \{\lambda \in E \mid \langle \lambda, \check{\alpha} \rangle\} \in \mathbb{Z}$ *for all* $\alpha \in \Phi\}$.

Proof. See Example Sheet. □

Definition. For $1 \le i \le l$, define $\omega_i \in E$ by $\langle \omega_i, \check{\alpha}_i \rangle = \delta_{ij}$, with $\{\omega_i\}$ the *fundamental weights with respect to* Δ.

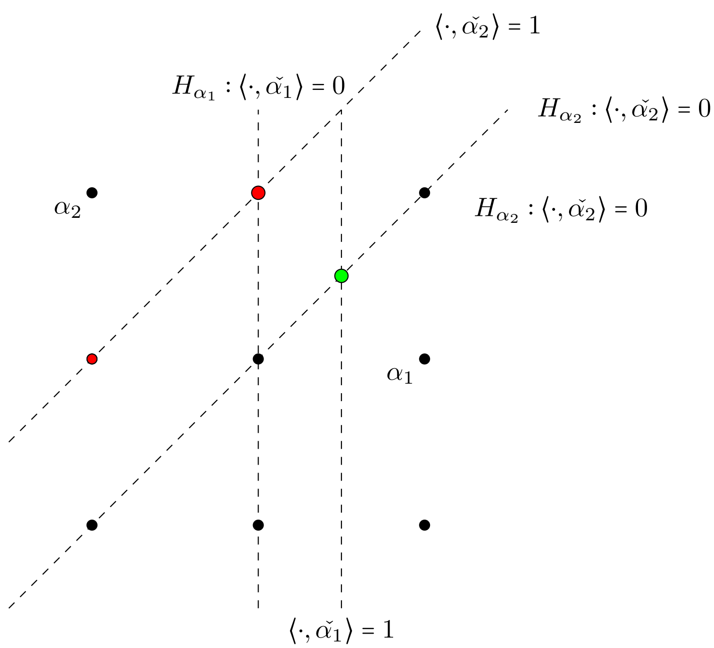
Then, Lemma 15.1 implies that $X = \{\sum c_i \omega_i \mid \; c_i \in \mathbb{Z}\}$.

Examples. 1. For A_2, ω_1 is the green point below, and ω_2 the red dot.

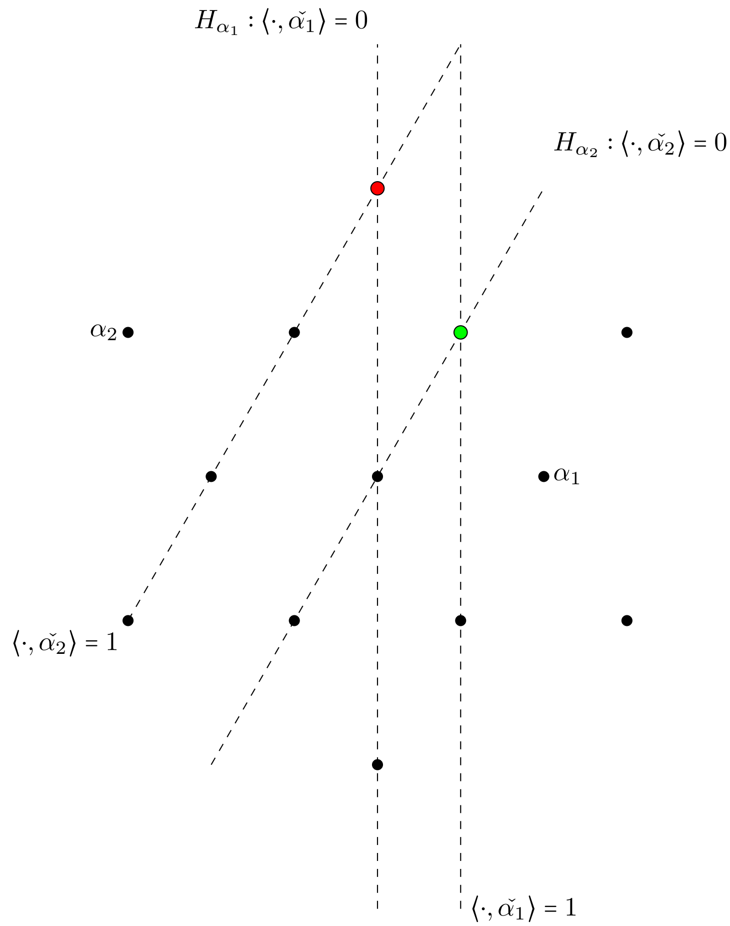


Here, $[X : \mathbb{Z}\Phi] = 3$

2. For B_2, again ω_1 is shown in green and ω_2 in red, and $[X : \mathbb{Z}\Phi] = 2$.



3. For G_2, again, ω_1 is in green, ω_2 in red, and now $X = \mathbb{Z}\Phi$.



Definition. $\lambda \in X$ is *dominant* if $\langle \lambda, \check{\alpha} \rangle \geq 0$ for all $\alpha \in \Phi^+$.

This is equivalent to:

- λ is in the closure of the fundamental Weyl chamber with respect to Δ.
- $\lambda = \sum_{i=1}^{l} c_i \omega_i$ with all $c_i \in \mathbb{Z}_{\geq 0}$.

From now on, $\mathfrak{g}$ is semisimple with root system Φ and Cartan subalgebra $\mathfrak{t}$. Choose $e_\alpha \in \mathfrak{g}_\alpha$ for each $\alpha \in \Phi$ such that $[e_\alpha, e_{-\alpha}] = h_\alpha$ and $\varphi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ a finite dimensional representation.

Lemma 14.2. $V = \bigoplus_{\lambda \in \mathfrak{t}^*} V_\lambda$ *where* $V_\lambda = \{v \in V \mid \ t \cdot v = \lambda(t) v \ \ \forall t \in \mathfrak{t}\}$.

Proof. This follows directly from Lemma 9.1, where the commuting semisimple endomorphisms are a basis of $\mathfrak{t}$. □

Proposition 14.3. *1. If $v \in V_\lambda$ then $e_\alpha \cdot v \in V_{\lambda+\alpha}$.*

2. If $V_\lambda \neq 0$ then $\lambda \in X$, i.e. $\lambda(h_\alpha) \in \mathbb{Z}$ for all α.

3. $\dim V_\lambda = \dim V_{w\lambda}$ for all $w \in W$.

Proof. 1. Fix $t \in \mathfrak{t}$. Then

$$t \cdot e_\alpha v = ([t, e_\alpha] + e_\alpha \cdot t)v = \alpha(t) e_\alpha(v) + e_\alpha \lambda(t) v = (\alpha + \lambda)(t) e_\alpha v.$$

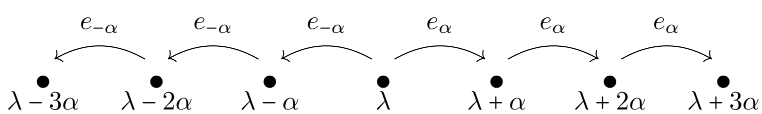
2. Consider $V|_{m_\alpha}$. We know h_α acts by integer weights, so $\lambda(h_\alpha) \in \mathbb{Z}$.

3. It suffices to assume $w = w_\alpha$ for some $\alpha \in \Phi$. Consider

$$V|_{m_\alpha} = \bigoplus V^j \text{ for } V^j \ m_\alpha\text{–irreducible representations}$$

Since the h_α–weight spaces of V^j are one dimensional we can choose a basis $v_1, \cdots, v_n$ for V_λ with each v_i in a distinct V^j.

It suffices to prove that given $v_i \in V^j$ there is an $x \in m_\alpha$ with $x \cdot v_i \in V_{w_\alpha \lambda}$. We have $w_\alpha(\lambda) = \lambda - \langle \lambda, \check{\alpha} \rangle \alpha$. We know that $\{e_{-\alpha}^k v_i, e_\alpha^k v_i \mid k \in \mathbb{Z}_{\geq 0}\}$ spans V^j.



Let $M = \max\{k \mid e_\alpha^k v_i \neq 0\}$, $m = \max\{k \mid e_{-\alpha}^k v_i \neq 0\}$. It suffices to show $-m \leq -\langle \lambda, \check{\alpha} \rangle \leq M$. But

$$(\lambda + M\alpha)(h_\alpha) = -(\lambda - m\alpha)(h_\alpha)$$

so $\lambda(h_\alpha) = m - M$, and since $\lambda(h_\alpha) = \langle \lambda, \check{\alpha} \rangle$ we are done.

□

Definition. $v \in V$ is a *highest weight vector* if

- $v \neq 0$.
- $v \in V_\lambda$ for some λ.
- $e_\alpha(v) = 0$ for all $\alpha \in \Phi^+$.

Example. On the example sheet, you show there is a root α_0 of maximal height with representationsect to Δ. Any nonzero $v \in \mathfrak{g}_{\alpha_0}$ is a highest weight vector with respect to the adjoint representation.

Warm up for Lecture 17: $\mathfrak{g} = \mathfrak{sl}_3$, $\mathfrak{t}$ a Cartan subalgebra with basis $h_{\alpha_1} = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}, h_{\alpha_2} = \begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix}$. Let V be the defining representation of $\mathfrak{g}$ with standard basis $\{e_1, e_2, e_3\}$. We would like $\lambda_i \in \mathfrak{t}^*$ such that $V = \bigoplus V_{\lambda_i}$, and a basis for each V_i.

$$h_{\alpha_1} \cdot e_1 = e_1; \quad h_{\alpha_1} \cdot e_2 = -e_2; \quad h_{\alpha_1} \cdot e_3 = 0.$$

$$h_{\alpha_2} \cdot e_1 = 0; \quad h_{\alpha_2} \cdot e_2 = e_2; \quad h_{\alpha_2} \cdot e_3 = -e_3,$$

so let $\lambda_1(h_{\alpha_1}) = 1, \lambda_1(h_{\alpha_2}) = 0$. Then $V_{\lambda_1} = \langle e_1 \rangle$.

Let $\lambda_2(h_{\alpha_1}) = -1, \lambda_2(h_{\alpha_2}) = 1$, then $V_{\lambda_2} = \langle e_2 \rangle$.

Let $\lambda_3(h_{\alpha_1}) = 0, \lambda_3(h_{\alpha_2}) = -1$, then $V_{\lambda_3} = \langle e_3 \rangle$.

Note: $\lambda_1 = \omega_1, \lambda_2 = -\omega_1 + \omega_2, \lambda_3 = -\omega_2$ in the weight diagram for A_2, and e_1 is a highest weight vector, since $e_\alpha \cdot v_\lambda \in V_{\lambda+\alpha}$, and $\omega_1 + \alpha_1$, $\omega_1 + \alpha_2$ are not weights for V.

Lemma 14.4. *1. V has a highest weight vector.*

2. If $v \in V_\lambda$ is a highest weight vector then λ is a dominant weight.

Proof. 1. Choose any nonzero $v_0 \in V_\lambda$ (any λ). If v_0 is a highest weight vector then done, otherwise choose $\alpha \in \Phi^+$ such that $e_\alpha v_0 \neq 0$. Let $k_1 = \max\{k \mid e_\alpha^k v_0 \neq 0\}$, and let $v_1 = e_\alpha^{k_1} v_0 \in V_{\lambda + k_1 \alpha}$. Repeat this argument, replacing v_0 by v_1. This process must end, since V is finite dimensional and each v_i is in a distinct weight space, as we always add on a positive root.

2. For $\alpha \in \Phi^+$, we need to show that $\langle, \lambda, \check{\alpha} \rangle \in \mathbb{Z}_{\geq 0}$. Consider $m_\alpha = \langle e_\alpha, h_\alpha, e_{-\alpha}$ acting on V. $e_\alpha \cdot v = 0 \quad h_\alpha \cdot v = \lambda(h_\alpha) v$ so v is a highest weight vector for any $m_\alpha \cong \mathfrak{sl}_2$ acting on V, and hence $\lambda(h_\alpha) \in \mathbb{Z}_{\geq 0}$.

□

Our aim for now is to show that there is a correspondence between finite dimensional irreducible representations of $\mathfrak{g}$ and dominant weights.

Universal Enveloping Algebras

Our motivation for the next section is the following: if V is a representation of $\mathfrak{g}$, $v \in V$ then $\cdots e_\alpha e_\beta e_\gamma e_\alpha \cdot v$ is not necessarily in V, but it will be in the algebra we are about to define.

Definition. Suppose V is a vector space over k. The *tensor algebra* of V is

$$\mathcal{T}(V) = k \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots = \bigoplus_{n \geq 0} V^{\otimes n},$$

with associated product given on generators by

$$v_1 \otimes \cdots \otimes v_k \cdot u_1 \otimes \cdots u_m = v_1 \otimes \cdots \otimes v_k \otimes u_1 \otimes \cdots u_m \in V^{\otimes (k+m)}.$$

The *symmetric algebra*

$$\mathrm{Sym}(V) = \mathcal{T}(V)/I$$

where I is the two-sided ideal generated by $\{x \otimes y - y \otimes x \mid x, y \in V\}$.

Note that:

- $\mathrm{Sym}(V) = \bigoplus_{n \geq 0} \mathrm{Sym}^n(V)$.
- We can identify $\mathrm{Sym}(V)$ with $k[V]$, the algebra of polynomials on V.
- $\mathcal{T}(V)$ and $\mathrm{Sym}(V)$ are graded.

Definition. If $\mathfrak{g}$ is a Lie algebra, the *universal enveloping algebra* of $\mathfrak{g}$ is

$$\mathcal{U}(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/J,$$

where J is the two-sided ideal generated by $\{x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g}\}$.

Note that:

- We often write $x \otimes y$ as xy.
- If V is a representation of $\mathfrak{g}$ then V is a $\mathcal{U}(\mathfrak{g})$-module via

  $$x_1 \otimes \cdots \otimes x_n \cdot v = x_1 \cdots x_n v.$$

  This is well-defined since $(x \otimes y - y \otimes x)v = (xy - yx)v = [xy]v$.
- Recall that if V is a finite dimensional representation of $\mathfrak{sl}_2$ we defined $\Omega = ef + fe + \frac{1}{2}h^2 \in \mathfrak{gl}(V)$. Ω is naturally an element of $\mathcal{U}(\mathfrak{g})$ independent of V.
- In general, if $\mathfrak{g}$ is semisimple with basis $\{x_1, \cdots x_n\}$ let $\{y_1, \cdots, y_n\}$ be the dual basis with respect to the Killing form. Then $\Omega = \sum_{i=1}^n x_i y_i \in \mathcal{U}(\mathfrak{g})$ is called a *Casimir element* (version 3). In fact, $\Omega \in Z(\mathcal{U}(\mathfrak{g}))$.
- $\mathcal{U}(\mathfrak{g})$ is not graded, e.g. $\mathfrak{g} \otimes \mathfrak{g}$ is not closed under addition, but it does have a filtration. Let

  $$U_n = \text{ image of } \bigoplus_{i=0}^n \mathfrak{g}^{\otimes i} \text{ in } \mathcal{U}(\mathfrak{g})$$

  Then $U_n U_m \subseteq U_{n+m}$.

  Exercise. If $x \in U_n, y \in U_m$ then $xy - yx \in U_{m+n-1}$.

  Let $x = \sum_{i=1}^k \lambda_i x_1^i \cdots x_n^i$, $y = \sum_{j=1}^l \mu_j y_1^j \cdots y_m^j$, so

  $$xy - yx = \sum_{i,j} \lambda_i \mu_j (x_1^i \cdots x_n^i y_1^j \cdots y_m^j - y_1^j \cdots y_m^j x_1^i \cdots x_n^i),$$

so it suffices to show $x_1 \cdots x_n y_1 \cdots y_m - y_1 \cdots y_m x_1 \cdots x_n \in U_{m+n-1}$ for $x_i, y_i \in \mathfrak{g}$. Now, $x_1 \cdots x_n y_1 \cdots y_m - y_1 \cdots y_m x_1 \cdots x_n$ can be written

$$\begin{gathered} x_1 \cdots x_n y_1 \cdots y_m - x_2 \cdots x_n y_1 \cdots y_m x_1 \\ + x_2 \cdots x_n y_1 \cdots y_m x_1 - x_3 \cdots x_n y_1 \cdots y_m x_1 x_2 \\ + \cdots \\ + x_n y_1 \cdots y_m x_1 \cdots x_{n-1} - y_1 \cdots y_m x_1 \cdots x_n . \end{gathered}$$

So, it suffices to show we can pull an element of $\mathfrak{g}$ from the front to the end of the string and get a difference in U_{m+n-1}, and then apply this to $x_1, x_2, \cdots, x_n$ in turn to show that each line above is in U_{n+m-1}. For x_1, we have

$$\begin{aligned} x_1 \cdots x_n y_1 \cdots y_m - x_2 \cdots y_m x_1 &= [x_1, x_2] x_3 \cdots x_n y_1 \cdots y_m \\ &+ x_2 [x_1, x_3] \cdots y_m \\ &+ x_2 x_3 [x_1, x_4] \cdots y_m \\ &+ \cdots \\ &+ x_2 \cdots y_{m-2} [x_1, y_{m-1}] y_m \\ &+ x_2 \cdots y_{m-1} [x_1, y_m] \end{aligned}$$

where each term is in U_{m+n-1} since $\mathfrak{g}$ is closed under the Lie bracket, and similarly for $x_2, \cdots, x_n$.

Definition. The *associated graded algebra* is

$$\operatorname{gr}(\mathcal{U}(\mathfrak{g})) = U_0 \oplus \left(\bigoplus_{n \geq 1} U_n / U_{n-1} \right).$$

Theorem. (Poincare-Birkhoff-Witt) There is an algebra isomorphism $\operatorname{Sym}(\mathfrak{g}) \cong \operatorname{gr}(U)$.

Proof. (Sketch) Defining a map $\mathfrak{g} \longrightarrow U_n \longrightarrow U_n / U_{n-1}$ gives a map from the tensor algebra to the associated graded algebra, which by the exercise above factors through the symmetric algebra $\operatorname{Sym}(\mathfrak{g})$.

$$\begin{array}{ccc} \mathcal{T}(\mathfrak{g}) & \longrightarrow & \operatorname{gr}(\mathcal{U}) \\ & \searrow & \uparrow \\ & & \operatorname{Sym}(\mathfrak{g}) \end{array}$$

Showing this map is surjective is straightforward, injective is harder. For a proof, see Humphrey's 17.4. □

Corollary 15.1. *If $\{x_1, \ldots, x_n\}$ is a basis of $\mathfrak{g}$ then $\{x_1^{k_1} \cdots x_n^{k_n} \mid k_i \in \mathbb{Z}_{\geq 0}\}$ is a basis for $\mathcal{U}(\mathfrak{g})$.*

Proof. A basis for Sym($\mathfrak{g}$) gives a basis for gr($\mathcal{U}$) which gives a basis for $\mathcal{U}(\mathfrak{g})$. □

Note that this implies that $\mathfrak{g}$ injects into $\mathcal{U}(\mathfrak{g})$.

Lemma 15.2. *If V is a representation of $\mathfrak{g}$ and $v \in V$ then the minimal subrepresentation of V containing v is*

$$\mathcal{U}(\mathfrak{g})v = \{uv \mid \; u \in \mathcal{U}(\mathfrak{g})\}.$$

Proof. It is straightforward to check that $\mathcal{U}(\mathfrak{g})$ contains:

- the elements $x_1 \cdots x_k v$ for all $x_i \in \mathfrak{g}$,
- all scalar multiples of the above, and
- sums of the above.

□

Warm up for lecture 18: Let V be a $\mathbb{C}$-vector space with basis $\{v_0, v_1, \cdots, \}$ and define an action of $\mathfrak{sl}_2$ ib V by

$$e \cdot v_0 = 0; \quad h \cdot v_0 = 0; \quad f \cdot v_i = v_{i+1} \;\; \forall i.$$

We wish to show that v_0, v_1 are highest weight vectors for the avtion of $\mathfrak{sl}_2$.

So, require $e \cdot v_j = 0$ for $j = 0, 1$. Done for $j = 0$. For $j = 1$,

$$e \cdot v_1 = ef \cdot v_0 = ([ef] + fe)v_0 = [ef]v_0 = hv_0 = 0.$$

We also need $\langle v_0 \rangle, \langle v_1 \rangle$ to contain their own images under h. We're done for v_0, and for v_1,

$$h \cdot v_1 = h \cdot fv_0 = ([hf] + fh)v_0 = [hf]v_0 = -2fv_0 = -2v_1$$

so $v_1 \in V_{-2}$ is a highest weight vector.

Note:

- $\langle \{v_1, v_2, \cdots\} \rangle = W$ is a subrepresentation of V with $V/W \cong V(0)$.
- In general, if $V^{(n)}$ is a vector space with basis $\{v_0, v_1, \cdots\}$ with $\mathfrak{sl}_2$ action given by
$$e \cdot v_0 = 0, \quad h \cdot v_0 = nv_0, \quad f \cdot v_i = v_{i+1},$$
then v_{n+1} is a highest weight vector and $W_n = \langle \{v_{n+1}, v_{n+2} \cdots\} \rangle$ has
$$V^{(n)}/W_n \cong V(n).$$

Highest Weight Modules

Throughout, $\mathfrak{g}$ is semisimple, $\mathfrak{t}$ is a Cartan subalgebra, Φ is roots of $\mathfrak{g}$ with respect to $\mathfrak{g}$ and $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ is a root basis.

Recall that if V is a representation of $\mathfrak{g}$, $V_\lambda = \{v \in V : tv = \lambda(t)v \forall t \in \mathfrak{t}^*\}$. Note:

- This definition makes sense even if V is infinite dimensional.
- The definition of highest-weight vector also makes sense if V is infinite dimensional.
- If $e_\alpha \in \mathfrak{g}_\alpha$ is non-zero then $e_\alpha \cdot V_\lambda \subseteq V_{\lambda+\alpha}$ (even if V is infinite dimensional).

Definition. A representation V of $\mathfrak{g}$ is called a *highest-weight module* if V contains a highest weight vector v such that $V = \mathcal{U}(\mathfrak{g})v$.

Examples. 1. Any finite dimensional irreducible representation V of $\mathfrak{g}$ is a highest weight module, since V contains a highest weight vector by Lemma 15.4, and Lemma 16.2 implies that $\mathcal{U}(\mathfrak{g})$ is a subrepresentation, so by Weyl's Theorem is all of $\mathfrak{g}$.

2. The representation of $\mathfrak{sl}_2$ from the warm up: v_0 highest weight vector, $v_i = f^i v_0$ so $V = \mathcal{U}(\mathfrak{g})v_0$.

Note:

- Not every highest weight module is irreducible.
- If V is an infinite dimensional highest weight module, $v \in V_\lambda$ a highest weight vector then λ is not necessarily dominant.

Notation: $\eta^+ = \sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$, $\eta^- = \sum_{\alpha\in\Phi^-} \mathfrak{g}_\alpha$ so $\mathfrak{g} = \eta^+ \oplus \mathfrak{t} \oplus \eta^-$

Example. With the usual root basis for $\mathfrak{sl}_n$, η^+ is the strictly upper triangular matrices.

Lemma 16.1. *Suppose V is a highest weight module with highest weight vector v such that $V = \mathcal{U}(\mathfrak{g})v$. Then $V = \mathcal{U}(\eta^-)v$.*

Proof. Choose a basis $\{x_1, \cdots x_n\}$ of η^-, $\{t_1, \cdots, t_l\}$ of $\mathfrak{t}$ and $\{y_1, \cdots y_n\}$ of η^+. Then $\mathcal{U}(\mathfrak{g}) = \langle x_1^{k_1} \cdots x_n^{k_n} t_1^{m_1} \cdots t_l^{m_l} y_1^{r_1} \cdots y_n^{r_n} v\rangle$ but $y_i \cdot v = 0$ for all i, and $t_i \cdot v \in \langle v\rangle$ so $\mathcal{U}(\mathfrak{g})v = \mathcal{U}(\eta^-)v$. □

Proposition 16.2. *Let V be a highest weight module with highest weight vector $v_\lambda \in V_\lambda$ such that $V = \mathcal{U}(\mathfrak{g})v_\lambda$. Then:*

1. *$V = \bigoplus_{\mu\in D(\lambda)} V_\mu$ where $D_\lambda = \{\lambda - \sum_{i=1}^l k_i\alpha_i \mid k_i \in \mathbb{Z}_{\geq 0}\}$.*
2. *Any submodule of V is a direct sum of weight spaces V_μ.*
3. *$Dim(V_\lambda) = 1$ and any other V_μ is finite dimensional.*

4. *V is irreducible if and only if every highest weight vector lies in* V_λ.

5. *V contains a maximal proper subrepresentation.*

Proof. 1. $V = \mathcal{U}(\eta^-)v_\lambda = \langle\{e_{-\beta_1}e_{-\beta_2}\cdots e_{-\beta_k}v_\lambda \mid \beta_i \in \Phi^+, k \in \mathbb{Z}_{\geq 0}\}\rangle$, with the generators in $V_{\lambda - \sum_{i=1}^k \beta_i}$

2. Exercise.

3. Follows form 1, plus the fact that given μ there is only a finite number of ways to write μ as $\lambda - \sum_{i=1}^k \beta_i$ for $\beta_i \in \Phi^+$.

4. Now suppose that V has a highest weight vector $v_\mu \in V_\mu$ with $\mu \neq \lambda$. Then $\mathcal{U}(\mathfrak{g})v_\mu$ is a subrepresentation and $v_\lambda \notin \mathcal{U}(\mathfrak{g})v_\mu$. This is since weights for $\mathcal{U}(\mathfrak{g})v_\mu$ are of the form $\mu - \sum k_i\alpha_i$ so $\mathcal{U}(\mathfrak{g})v_\mu$ is a nontrivial proper subrepresentation.

   Suppose V is reducible with nontrivial proper subrepresentation U. By (2), U is a direct sum of weight spaces V_μ. Choose μ to be $\lambda = \mu - \sum k_i\alpha_i$ such that $V_\mu \subseteq U$ and $\sum k_i$ is minimal. Let $v_\mu \in V_\mu$ be nonzero, $\alpha \in \Phi^+$ and $e_\alpha \in \mathfrak{g}_\alpha$. Then
$$e_\alpha \cdot v_\mu \in V_{\mu+\alpha} \cap U = 0.$$
   by choice of U. So v_μ is a highest weight vector for V.

5. Take the sum of all the proper subrepresentations, call it $\mathcal{V}$. $v_\lambda \notin \mathcal{V}$ so $\mathcal{V} \neq V$.

□

Warm Up for Lecture 19: If $\Phi, \alpha \in \Delta$ and λ is a dominant weight, then $\lambda - \alpha$ is a dominant weight. In particular, $(\lambda-\alpha, \lambda-\alpha) \leq (\lambda, \lambda)$. This is because

$$(\lambda - \alpha, \lambda - \alpha) = (\lambda, \lambda) - (\alpha, \lambda) - (\lambda - \alpha, \alpha) = (\lambda, \lambda) - (\text{something positive}).$$

Definition. If V is a highest weight vector with $v \in V_\lambda$ and $V = \mathcal{U}(\mathfrak{g})v$, v a highest weight vector, say that V is *of highest weight* λ.

Fix a basis for $\mathfrak{g}$ of the form $\{h_{\alpha_i}, e_{\alpha_i} \mid 1 \leq i \leq l, \alpha \in \Phi\}$ such that $e_\alpha \in \mathfrak{g}_\alpha$, $[e_\alpha, e_{-\alpha}] = h_\alpha$.

Definition. Given $\lambda \in \mathfrak{t}^*$, the *Verma module* $M(\lambda)$ is

$$M(\lambda) = \mathcal{U}(\mathfrak{g})/K_\lambda$$

where K_λ is the left ideal generated by $\{e_\alpha, h_{\alpha_i} - \lambda(h_{\alpha_i}) \mid 1 \leq i \leq l, \alpha \in \Phi^+\}$.

Proposition 16.3. 1. *$M(\lambda)$ is a highest weight module of highest weight λ.*

2. *$M(\lambda)$ is universal in the sense that if $m_\lambda \in M(\lambda)_\lambda$ is a highest weight vector and V is a highest weight module of highest weight λ with highest weight vector $v_\lambda \in V_\lambda$ then there is a unique $\mathfrak{g}$-equivariant linear map $M(\lambda) \longrightarrow V$ with $m_\lambda \longmapsto v_\lambda$.*

Proof. 1. Let $m_\lambda = 1 + K_\lambda \in M(\lambda)$. Then

$$h_{\alpha_i}(m_\lambda) = h_{\alpha_i} + K_\lambda = \lambda(h_{\alpha_i})m_\lambda.$$

If $\alpha \in \Phi^+$ then

$$e_\alpha \cdot m_\lambda = e_\alpha + K_\lambda = K_\lambda = 0 \in M(\lambda),$$

so m_λ is a highest weight vector of highest weight λ. It is clear that $\mathcal{U}(\mathfrak{g})m_\lambda = M(\lambda)$ so any other highest weight vector is a scalar multiple of this one (by Proposition 17.2).

2. Note that $\{e_{-\beta_1}, \cdots e_{-\beta_k} \cdot m_\lambda \mid \beta_i \in \Phi^+, k \in \mathbb{Z}_{\geq 0}\}$ is a basis for $M(\lambda)$. Define $\varphi : M(\lambda) \longrightarrow V$ by $\varphi(e_{-\beta_1}, \cdots e_{-\beta_k} \cdot m_\lambda) = e_{-\beta_1}, \cdots e_{-\beta_k} \cdot v_\lambda$. Can check that φ is $\mathfrak{g}$-equivariant.

   Exercise. Check: if $\alpha \in \Phi^+$, $\varphi(e_\alpha e_{-\beta_1} \cdots e_{-\beta_k} m_\lambda) = e_\alpha e_{-\beta_1} \cdots e_{-\beta_k} v_\lambda$ (by induction on k, using bracket).

□

Proposition 16.4. *Given $\lambda \in \mathfrak{t}^*$, there is a unique irreducible highest weight module with highest weight λ, called $V(\lambda)$.*

Proof. By Proposition 17.2, $M(\lambda)$ has a unique maximal proper submodule I. Then $M(\lambda)/I$ is irreducible, and uniqueness follows form the universal property. □

Example. In warm up for lecture 18, $V = M(0), I = \langle, \{v_1, v_2, \cdots\}\rangle$ and $M(0)/I \cong V(0)$ the trivial representation of $\mathfrak{sl}_2$.

Proposition 16.5. *$V(\lambda)$ is finite dimensional if and only if λ is a dominant weight.*

Proof. If $V(\lambda)$ is finite dimensional then use Lemma 15.4 (ii) by acting by the root $\mathfrak{sl}_2$s.

Now suppose that λ is a dominant weight. We proceed by a series of reductions:

1. Let $\Pi(\lambda) = \{\mu : V(\lambda)_\mu \neq 0\}$. It suffices to show that $\Pi(\lambda)$ is finite, by Proposition 17.2 (3).
2. It suffices to show that $\dim(V(\lambda)_\mu) = \dim(V(\lambda)_{w\mu})$ for all elements w of the Weyl group, since there are only finitely many dominant weights in $\Pi(\lambda)$ and any weight is conjugate under the Weyl group to a dominant weight.
3. It suffices to show (2) for the simple reflections.

4. It suffices to show that for fixed $\mu \in \Pi(\lambda)$, $\alpha_i \in \Delta$,

$$V(\lambda)_\mu \subseteq \text{ finite dimensional } m_{\alpha_i} \text{ subrepresentation of } V(\lambda),$$

by the same logic as Proposition 15.3.

Fix such a μ, α_i.

Claim: There is an $M \geq 0$ such that $V(\lambda)_{\mu+n\alpha_j} = 0$ for all $n > M$.

Proof of Claim: We know $\mu = \lambda - \sum k_i \alpha_i$, with $k_i \in \mathbb{Z}_{\geq 0}$. Take $M = k_j$.

Now, we know $V(\lambda)_\mu \subseteq \bigoplus_{n \leq m} V(\lambda)_{\mu+n\alpha_j}$, which a priori is a representation of m_{α_j} but isn't finite dimensional.

It's enough to show that given $v \in V(\lambda)_\mu$ there is an $n \geq 0$ such that $e_{-\alpha_j}^n v = 0$ Take v nonzero in $V(\lambda)_\mu$. We know

$$v = e_{-\beta} e_{-\beta_2} \cdots e_{-\beta_k} v_\lambda,$$

where $v_\lambda \in V(\lambda)_\lambda$, for some $\beta_i \in \Phi^+, k \in \mathbb{Z}_{\geq 0}$.

Now we proceed by induction on k: for $k = 0$, $v = v_\lambda$ we have

$$h_{\alpha_j} \cdot v = \lambda(h_{\alpha_j}) v.$$

Recall Lemma 4.1 says that for all $n \geq 1$

$$e_{\alpha_j} e_{-\alpha_j}^n v_\lambda = n(\lambda(h_{\alpha_j}) - n + 1) e_{-\alpha_j}^{n-1} v_\lambda.$$

Let $N = \lambda(h_{\alpha_j}) + 1 \geq 1$. Then we claim $e_{\alpha_j}^N v_\lambda$ is a highest weight vector for $V(\lambda)$ if it is nonzero. Take $\alpha \in \Phi^+$, $e_{\alpha_j} e_{-\alpha_j}^N v_\lambda \in V(\lambda)_{\lambda - N\alpha_j + \alpha}$ so $e_{-\alpha_j}^N v_\lambda$ is a highest weight vector in $V(\lambda)_{\lambda - N\alpha_j} \neq V(\lambda)_\lambda$, which is a contradiction, so $e_{-\alpha_j}^N v_j = 0$.

The induction step follows from

Exercise. If $\alpha, \beta \in \Phi$ then

$$e_\alpha^n e_\beta = \sum_{i=1}^{A} c_i e_{\beta + i\alpha} e_\alpha^{n-i}$$

where the c_i are constants and $A = \max\{k : \beta + k\alpha \in \Phi\}$.

□

We've just shown that there is a bijection between dominant weights λ and finite dimensional irreducible representations of $\mathfrak{g}$ $V(\lambda)$.

Weights and Characters

Throughout this section, $\mathfrak{g}$ is a semisimple Lie algebra, $\mathfrak{t}$ a Cartan subalgebra, and $\Delta = \{\alpha_1, \cdots, \alpha_l\} \subseteq \Phi$ is a choice of root basis inside the set of roots, X is the weight lattice, and W is the Weyl group.

Warm up for Lecture 20: Let $\rho = \frac{1}{2}\sum_{\gamma \in \Phi^+} \gamma$. For A_1, A_2, B_2 we compute $\langle \rho, \check{\alpha} \rangle$ for all $\alpha \in \Delta$. For A_2, B_2 with $\Delta = (\alpha_1, \alpha_2), \rho = \frac{\alpha_1 + \alpha_2}{2} = \omega_1 + \omega_2$.

- A_1: $\{-\alpha, \alpha\}, \rho = \frac{\alpha}{2}$, $\langle \rho, \check{\alpha} \rangle = 1$.
- A_2: $\langle \rho, \check{\alpha_1} \rangle = \langle \omega_1 + \omega_2, \check{\alpha_1} \rangle = 1$.
- B_2: $\langle \alpha_i, \check{\rho} \rangle = 1$.

Claim: $\rho = \sum_{i=1}^{l} \omega_i$.

Proof: It suffices to show that $\langle \rho, \check{\alpha_j} \rangle = 1$ for all $\alpha_j \in \Delta$. We have

$$w_{\alpha_j}(\rho) = \rho - \langle \rho, \check{\alpha_j} \rangle \alpha_j,$$

and also

$$w_{\alpha_j}(\rho) = w_{\alpha_j}(\frac{1}{2}(\sum_{\alpha \in \Phi^+ \setminus \{\alpha_j\}} \alpha) + \frac{1}{2}\alpha_j),$$

and w_{α_j} permutes $\Phi^+ \setminus \{\alpha_j\}$ so this is

$$\frac{1}{2}\left(\sum_{\alpha \in \Phi^+ \setminus \{\alpha_j\}} \alpha\right) - \frac{1}{2}\alpha_j = \rho - \alpha_j.$$

So, comparing the two expressions, we obtain $\langle \rho, \check{\alpha_j} \rangle = 1$. □

Recall

$$\Pi(\lambda) = \{\mu \mid V(\lambda)_\mu \neq 0\}$$

Our goals for this section will be to answer the following questions:

1. What is $\Pi(\lambda)$?
2. What is $\dim(V(\lambda))$?

Definition. Define a partial ordering $\prec$ on X by

$$\mu \prec \lambda \iff \lambda - \mu = \sum_{i=1}^{l} k_i \alpha_i, \quad k_i \in \mathbb{Z}_{\geq 0} \forall i$$

Note that:

- $\Pi(\lambda) \subseteq \{\mu : \mu \prec \lambda\}$
- To find $\Pi(\lambda)$, it's enough to find the dominant weights in $\Pi(\lambda)$.

Proposition 17.1. *Suppose λ, μ are both dominant weights, then*

$$\mu \in \Pi(\lambda) \iff \mu \prec \lambda$$

Proof. Suppose $\mu \prec \lambda$. We know that $\mu = \lambda - \sum_{\alpha\in\Phi^+} k_\alpha \alpha$ for some $k_\alpha \in \mathbb{Z}_{\geq 0}$, and we'll proceed by induction on $\sum k_\alpha$.

Base case: $\sum k_\alpha = 0$ (done).

Warm up case: Suppose $\mu = \lambda - \alpha$ for $\alpha \in \Phi^+$. Then

$$\langle \mu, \check{\alpha} \rangle = \langle \lambda, \check{\alpha} \rangle - 2 \geq 0$$

so $\langle \lambda, \check{\alpha} \rangle \geq 2$. Take $v_\lambda \in V_\lambda$ to be nonzero. Since $h_\alpha \cdot v_\lambda = n v_\lambda$ for some $n \geq 2$ we know that $e_{-\alpha} v_\lambda \neq 0$ by $\mathfrak{sl}_2$ theory (it's in $V(\lambda)_{\lambda-\alpha} = V(\lambda)_\mu$). Suppose we know the claim is true for $\sum k_\alpha = n-1$ and suppose $\sum k_\alpha = n$, so $\mu = \lambda - \beta_1 - \cdots - \beta_n$.

We now split into cases:

Case 1: $\langle \beta_i, \check{\beta_j} \rangle < 0$ for some i,j with $i \neq j$, without loss of generality $i < j$. This implies $\beta_i + \beta_j$ is a positive root (Weyl group + sum of positive roots is positive) so

$$\sum_{k=1}^{n} \beta_k = \sum_{k=1}^{i-1} \beta_i + \sum_{k=i+1}^{j-1} \beta_k + (\beta_i + \beta_j) = \sum_{k=1}^{n-1} \gamma_k$$

with $\gamma_k = \begin{cases} \beta_k & k \leq i-1 \\ \beta_{k+1} & i < k < j \\ \beta_{k+2} & j < k \\ \beta_i + \beta_j & k = n-1 \end{cases}$ and so $\gamma)k \in \Phi^+$ for all k and by the inductive hypothesis we are done.

Case 2: $\langle \beta_i, \check{\beta_j} \rangle \geq 0$ for all i,j with $i \neq j$.

In this case,

Claim: $\lambda - \sum_{i=1}^{r} \beta_i \in \Pi(\lambda)$ for all $1 \leq r \leq n$.

We'll prove the claim by induction on r. If $r = 1$,

$$0 \leq \langle \lambda - \sum_{i=1}^{n} \beta_i, \check{\beta_1} \rangle = \langle \lambda, \check{\beta_1} \rangle - 2 - \sum_{i=2}^{n} \langle \beta_i \check{\beta_1} \rangle$$

The leftmost term is positive by assumption, so $\langle \lambda, \check{\beta_1} \rangle \geq 2$, so considering the action of m_{β_1} so $\lambda - \beta_1 \in \Pi(\lambda)$.

For the induction step, the same logic implies that

$$\langle \lambda - sum_{i=1}^{r} \beta_i, \check{\beta_r} \geq 0$$

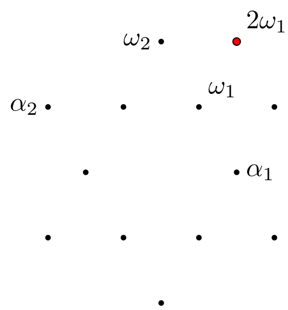
and the action by m_{β_r} so $\lambda - \sum_{i=1}^{r} \beta_i \in \Pi(\lambda)$. □

Warm Up for Lecture 21: Let $\{\alpha_1, \alpha_2\}$ be root basis of G_2 with α_1 short. We wish to calculate $\Pi(2\omega_1)$.

ω_1 has

$$\langle \omega_1, \check{\alpha_2} \rangle = 0, \langle \omega_1, \check{\alpha_1} \rangle = 1$$

and the dominant weights in $\Pi(2\omega_1)$ are exactly the dominant μ with $\mu \prec 2\omega_1$ by Proposition 18.1.



$2\omega_1 = 2(\alpha_2 + 2\alpha_1) = 2\alpha_2 + 4\alpha_1$.

$\omega_2 = 3\alpha_1 + 2\alpha_2$ so $\omega_2 \prec 2\omega_1$, but $\omega_2 + \omega_1 \nprec 2\omega_1$. So the dominant weights in $\Pi(2\omega_1)$ are $\{\omega_1, 2\omega_1, \omega_2, 0\}$. The Weyl conjugates of ω_1 are the short roots, and of the Weyl conjugates of ω_2 are all the long roots. So,

$$\begin{aligned}\Pi(2\omega_1) &= \{\text{ short roots }\} \cup \{\text{ 2(short roots)}\} \cup \{\text{ long roots }\} \\ &= \Phi \cup \{\pm 2\omega_1, \pm 2\alpha_1, \pm 2(\alpha_1 + \alpha_2), 0\}\end{aligned}$$

Definition. Let $\mathbb{Z}[X]$ be the free $\mathbb{Z}$-module with basis $\{e^\mu \mid \mu \in X\}$ with multiplication $e^\mu e^\lambda = e^{\lambda+\mu}$, commutative ring (extending linearly) with identity $e^0 = 1$.

Definition. Let V be a finite dimensional representation of $\mathfrak{g}$. The formal character of V is

$$ch(V) = \sum_{\mu \in X} \dim(V_\mu) e^\mu \in \mathbb{Z}[X]$$

(note that the sum is finite).

Recall form the example sheets that $l(w)$ is the minimal n such that w can be written as a product of n simple reflections.

Definition. The *sign* of w, $sn(w) = (-1)^{l(w)}$.

Example. For $\mathfrak{g} = \mathfrak{sl}_n$, $sn(w) = sgn(w) \in S_n$ as $W \cong S_n$.

Theorem. (Weyl Character Formula) If λ is a dominant weight, and $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha = \sum_{j=1}^n \omega_j$,

$$ch(V(\lambda)) = \frac{\sum_{w \in W} sn(w) e^{w(\lambda+\rho)}}{e^\rho \prod_{\alpha \in \Phi^+}(1 - e^{-\alpha})}$$

Proof. c.f. Grojnowski Chapter 10. □

Corollary. (Weyl Denominator Formula)

$$e^\rho \prod_{\alpha \in \Phi^+}(1 - e^{-\alpha}) = \sum_{w \in W} sn(w) e^{w\rho}$$

Proof. $ch(V(0)) = 1$, so plug $\lambda = 0$ into the character formula. □

Corollary. (Weyl Dimension Formula) If λ is a dominant weight then

$$\dim(V(\lambda)) = \frac{\prod_{\alpha\in\Phi^+}\langle\lambda+\rho,\check{\alpha}\rangle}{\prod_{\alpha\in\Phi^+}\langle\rho,\check{\alpha}\rangle}$$

Proof. By definition, $ch(V(\lambda)) = \sum_{\mu\in X}\dim(V_\mu)e^\mu$. We'd like to substitute $e^\mu = 1$ into the character formula for any μ, but could get $\frac{0}{0}$.

Suppose $\mu \in X$, $p \in \mathbb{Z}[X]$. Define $f_\mu(p) : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ by

$$f_\mu(e^\lambda)(t) = t^{-(\mu,\lambda)}$$

and extending linearly.

Then f_μ is multiplicative and $f_\mu(p)$ is a continuous differentiable function on $\mathbb{R}_{>0}$. Apply to the denominator formula so for all $\mu \in X$, $t \in \mathbb{R}_{>0}$,

$$t^{-(\rho,\mu)}\prod_{\alpha\in\Phi^+}(1-t^{(\alpha,\mu)}) = \sum_{w\in W} sn(w)t^{-(\rho,\omega\alpha)}$$

And also pplying f_p to the Weyl character formula,

$$f_p(ch(V(\lambda)))(t) = \frac{\sum_{w\in W} t^{-(p,w(\lambda+\rho))}}{t^{-(p,\rho)}\prod_{\alpha\in\Phi^+}(1-t^{(p,\alpha)})}$$

Using the above with $\mu = \lambda + \rho$,

$$f_p(ch(V(\lambda)))(t) = \frac{t^{(-p,\lambda+\rho)}\prod_{\alpha\in\Phi^+}(1-t^{-(\alpha,\lambda+\rho)})}{t^{-(p,\rho)}\prod_{\alpha\in\Phi^+}(1-t^{(p,\alpha)})}$$

Note that

$$f_p(ch(V(\lambda)))(t) = \sum_{\mu\in X}\dim(V(\lambda))t^{-(p,\mu)}$$

so taking a limit as t tends to 1,

$$\dim(V(\lambda) = \frac{\prod_{\alpha\in\Phi^+}(\lambda+\rho,\alpha)}{\prod_{\alpha\in\Phi^+}(\rho,\alpha)} = \frac{\prod_{\alpha\in\Phi^+}\langle\lambda+\rho,\check{\alpha}\rangle}{\prod_{\alpha\in\Phi^+}\langle\rho,\check{\alpha}\rangle}$$

□

Example. Let $\mathfrak{g} = \mathfrak{sl}_3$ and let $\lambda = m_1\omega_1 + m_2\omega_2$. We will calculate $\dim(V(\lambda))$.

$\rho = \alpha_1 + \alpha_2$ and so

$$\begin{aligned}\prod_{\alpha\in\Phi^+}\langle\lambda+\rho,\check{\alpha}\rangle =& \langle m_1\omega_1 + m_2\omega_2 + \alpha_1 + \alpha_2, \check{\alpha}_1\rangle \\ & \times \langle m_1\omega_1 + m_2\omega_2 + \alpha_1 + \alpha_2, \check{\alpha}_2\rangle \\ & \times \langle m_1\alpha_1 + m_2\alpha_2 + \alpha_1 + \alpha_2, (\alpha_1 \check{+} \alpha_2)\rangle \\ =& (m_1+1)(m_2+1)(m_1+m_2+2)\end{aligned}$$

$\prod_{\alpha\in\Phi^+}\langle\rho,\check{\alpha}\rangle = (1)(1)(2)$ and therefore

$$\dim(V(\lambda)) = \frac{(m_1+1)(m_2+1)(m_1+m_2+2)}{2}.$$

Warm Up for lecture 22: For $\mathfrak{g} = \mathfrak{sl}_3$, with $V(\omega_1)$ the defining representation, decompose $\mathrm{Sym}^2(V(\omega_1))$ into irreducibles. Note that $\mathbf{e}_1$ is a highest weight vector for $V(\omega_1)$, and recall that a basis for $\mathrm{Sym}^2 V(\omega_1)$ is $\mathbf{e}_i \otimes \mathbf{e}_j$ with $1 \le i \le j \le 3$. For $\mathrm{Sym}^2(V(\omega_1))$, $\mathbf{e}_1 \otimes \mathbf{e}_1$ is a highest weight vector, since if $\alpha \in \Phi^+$,

$$e_\alpha \cdot (\mathbf{e}_1 \otimes \mathbf{e}_1) = (e_\alpha \mathbf{e}_1) \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes (e_\alpha \mathbf{e}_1) = 0 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes 0 = 0.$$

and if $t \in \mathfrak{t}$,

$$t \cdot \mathbf{e}_1 \otimes \mathbf{e}_1 = (t\mathbf{e}_1) \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes (t\mathbf{e}_1) = (\omega_1(t)\mathbf{e}_1) \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes (\omega_1(t)\mathbf{e}_1) = 2\omega_1(t)(\mathbf{e}_1 \otimes \mathbf{e}_1)$$

So $V(2\omega_1)$ is a subrepresentation.

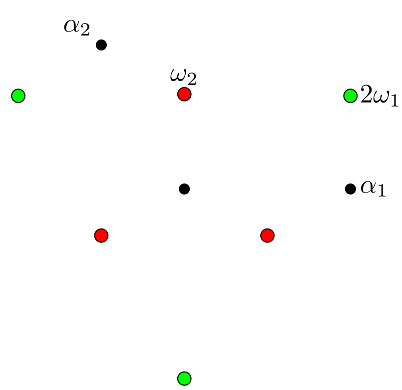
$$\dim(V(2\omega_1)) = \frac{1}{2}(3)(1)(4) = 6$$

so $\mathrm{Sym}^2 V(\omega_1) \cong V(2\omega_1)$.

Weight Diagrams for $\mathfrak{sl}_3$

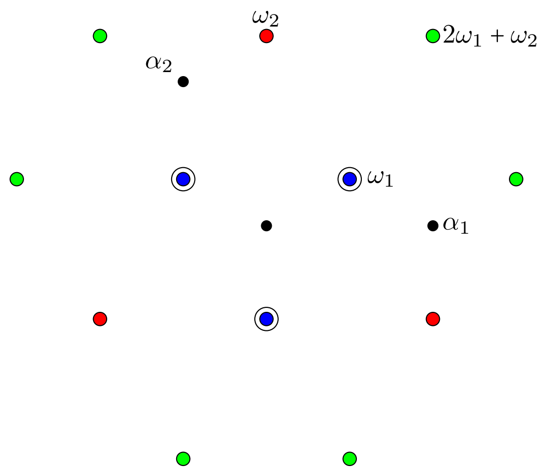
$V(2\omega_1)$

The dominant weights are $2\omega_1$ and $2\omega_1 - \alpha_1 = \omega_2$



$V(2\omega_1 + \omega_2)$

μ	Number of Weyl conjugates	$\dim(V(2\omega_1+\omega_2)_\mu)$
$2\omega_1 + \omega_2$	6	1
$2\omega_2$	3	1
ω_1	3	2



and $\dim(V(2\omega_1+\omega_2)) = \frac{1}{2}(3)(2)(5) = 15$.

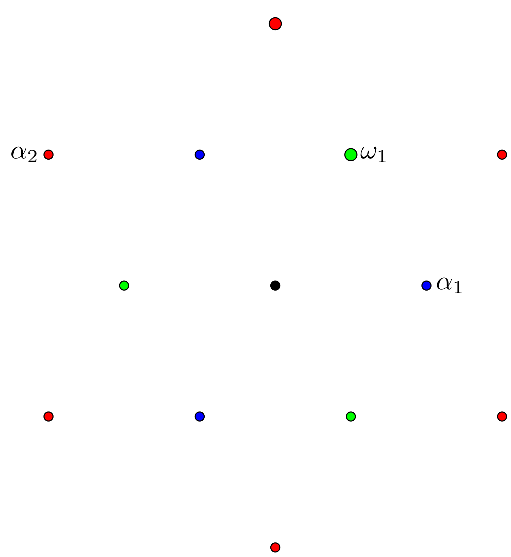
Handy fact: $\dim(V(\lambda)_\mu) \leq |\{\beta_1,\cdots,\beta_k \mid \mu = \beta_1 - \cdots - \beta_k\}$. This is since if $v \in V(\lambda)_\lambda$ is a highest weight vector then can find a basis for $V(\lambda)$ of the form $e_{-\beta_1}\cdots e_{-\beta_k}v$.

In our case, $\omega_1 = 2\omega_1+\omega_2-(\alpha_1+\alpha_2) = 2\omega_1+\omega_2-(\alpha_1)-(\alpha_2)$

Exercise. For $\mathfrak{g} = \mathfrak{sl}_3$, decompose $V(2\omega_1)\otimes V(\omega_2)$ into irreducibles.

Hint: $V(2\omega_1+\omega_2)$ is a subrepresentation.

Weight diagrams for G_2, adjoint representation



The long roots span an A_2 - can decompose the adjoint representation of G_2 under action of $\mathfrak{sl}_3$.

9 Groups

Throughout, $\mathfrak{g}$ semisimple, $\Delta, \Phi\mathfrak{t}$ as usual, $h_\alpha \in \mathfrak{t}$. For $\alpha \in \Phi$, choose $e_\alpha \in \mathfrak{g}_\alpha$ such that $[e_\alpha, e_{-\alpha}] = h_\alpha$.

We've defined two groups in this course:

1. The group of inner automorphisms of $\mathfrak{g}$:

$$G_{\mathrm{ad}} = \langle \exp(\operatorname{ad} x) \mid \operatorname{ad} x, x \in \mathfrak{g} \text{ is nilpotent} \rangle$$

2. The Weyl group W.

We can say some things about the structure of G_{ad}.

Example. $\mathfrak{g} = \mathfrak{sl}_2$

Recall that if $x \in \mathfrak{g}$ is nilpotent then $\exp(\operatorname{ad}(x))(y) = \exp(x) y \exp(x)^{-1}$ for all $y \in \mathfrak{g}$ (Lemma 8.1).

Claim 1: If $g \in \mathrm{SL}_2(\mathbb{C})$ then the map $\varphi_g : \mathfrak{g} \longrightarrow \mathfrak{g}$ given by

$$\varphi_g(x) = gxg^{-1}$$

is an element of G_{ad}.

To see this, let $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ as usual. Then te, tf are nilpotent for any $t \in \mathbb{C}$, and $\exp(te) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \exp(tf) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$ and

$$\mathrm{SL}_2(\mathbb{C}) = \langle \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \mid t \in \mathbb{C} \rangle \quad \square$$

Define $\varphi : \mathrm{SL}_2(\mathbb{C}) \longrightarrow G_{\mathrm{ad}}$ by $\varphi(g) = \varphi_g$. By Lemma 8.1 this is surjective.

$$\operatorname{Ker}(\varphi) = \langle \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \rangle$$

so $G_{\mathrm{ad}} = P\,\mathrm{GL}_2(\mathbb{C})$.

Now, for general $\mathfrak{g}$, given $\alpha \in \Phi$ define a map $u_\alpha : \mathbb{C} \longrightarrow G_{\mathrm{ad}}$ by

$$u_\alpha(t) = \exp(\operatorname{ad}(te_\alpha))$$

Lemma 18.1. *u_α is an injective homomorphism.*

Proof. To see that it is a homomorphism,

$$\exp(\operatorname{ad}(t+s)e_\alpha) = \exp(\operatorname{ad}(te_\alpha) + \operatorname{ad}(se_\alpha)) = \exp(\operatorname{ad}(te_\alpha)) \exp(\operatorname{ad}(se_\alpha))$$

To see that it is injective,

$$
\begin{aligned}
\exp(\mathrm{ad}(te_\alpha))h_\alpha =& h_\alpha + t[e_\alpha, h_\alpha] + \cdots \\
=& h_\alpha - 2te_\alpha \\
=& h_\alpha \iff t = 0
\end{aligned}
$$

□

Definition. u_α is called a *root group homomorphism.* Let U_α be the image of u_α, then U_α is called a *root group.*

Definition. If G is a group, a *representation* of G is a homomorphism $G \longrightarrow \mathrm{GL}(V)$ for some vector space V. All vocabulary translates over from representations already discussed.

Let $G_\alpha = \langle U_\alpha, U)-\alpha \rangle$, a subgroup of G_{ad}. Then $\mathfrak{g}$ is a representation of G_α

Lemma 18.2. *If V is an m_α subrepresentation of G, then V is a G_α subrepresentation of $\mathfrak{g}$.*

Proof. Take $v \in V$, $u_\alpha(t) \cdot v = \sum_{n=0}^{\infty} c_n e_\alpha^n \cdot v \in V$ so $G_\alpha \cdot V \subseteq V$. □

So:

- m_α is a G_α subrepresentation of $\mathfrak{g}$.
- $\mathrm{Ker}(\alpha) \subseteq \mathfrak{t}$ is a G_α subrepresentation of $\mathfrak{g}$.
- If $\beta \neq \pm\alpha$, then let
$$V_\beta = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\beta + k\alpha}$$
This is also a G_α subrepresentation.

Proposition 18.3. *1. There is a surjective homomorphism $G_\alpha \longrightarrow P\,\mathrm{GL}_2(\mathbb{C})$.*

2. There is a surjective homomorphism $\varphi : \mathrm{SL}_2(\mathbb{C}) \longrightarrow G_\alpha$ such that

$$\varphi(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}) = u_\alpha(t); \quad \varphi(\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}) = u_{-\alpha}(t)$$

3. G_α is isomorphic to either $P\,\mathrm{GL}_2(\mathbb{C})$ or $\mathrm{SL}_2(\mathbb{C})$.

Proof. Let $Inn(\mathfrak{sl}_2)$ be the group of inner automorphisms of $\mathfrak{sl}_2$, isomorphic to $P\,\mathrm{GL}_2(\mathbb{C})$.

1. Since m_α is a subrepresentation, the action of G_α preserves m_α. $u_\alpha(t), u_{-\alpha}(t)$ act on m_α by inner automorphisms, so we get a homomorphism
$$G_\alpha \longrightarrow Inn(\mathfrak{sl}_2) \cong P\,\mathrm{GL}_2(\mathbb{C})$$
This is surjective since if we write $m_\alpha \longrightarrow \mathfrak{sl}_2$ with $e_\alpha \longmapsto e$, $e_{-\alpha} \longmapsto f$ then with respect to this identification
$$u_\alpha(t)\,|_{m_\alpha} = \exp(\mathrm{ad}\,te); \quad u_{-\alpha}(t)\,|_{m_\alpha} = \exp(\mathrm{ad}\,tf)$$
and $Inn(\mathfrak{sl}_2) = \langle \exp(\mathrm{ad}\,te), \exp(\mathrm{ad}\,tf) \mid t \in \mathbb{C} \rangle$.

Note (1) and (2) imply 3 since $\mathrm{SL}_2(\mathbb{C}) \longrightarrow G_\alpha \longrightarrow P\,\mathrm{GL}_2 C$ where Π is the surjective map $\mathrm{SL}_2(\mathbb{C}) \longrightarrow P\,\mathrm{GL}_2(\mathbb{C})$ with kernel $\langle \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \rangle$.

For (2), the idea is to define a map $\varphi_\alpha : \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}(\mathfrak{g})$ and show that $im(\varphi_\alpha) = G_\alpha$.

Recall that $\mathfrak{g} = m_\alpha \oplus \mathrm{Ker}(\alpha) \oplus V_\beta$ where $V_\beta = \oplus_k \mathfrak{g}_{\beta+k\alpha}$ so it suffices to define an action on each piece.

m_α: $\varphi_\alpha(x)(y) = xyx^{-1}, x \in \mathrm{SL}_2, y \in m_\alpha$.

$\mathrm{Ker}(\alpha)$: Suppose $h \in \mathrm{Ker}(\alpha)$.

$$\begin{aligned} u_\alpha(t) &= \exp(\mathrm{ad}\, te_\alpha)(h) \\ &= h + t[e_\alpha, h] + \cdots \\ &= h \end{aligned}$$

so $u_\alpha(t)$ acts as the identity n $\mathrm{Ker}(\alpha)$. Similarly, $u_{-\alpha}(t)(h) = h$, so define $\varphi_\alpha(x)(h) = h$ for all $x \in \mathrm{SL}_2$, $h \in \mathrm{Ker}(\alpha)$.

V_β: This uses

Warm up for Lecture 24: Let $P_n = \{a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n \mid a_i \in \mathbb{C}\}$. Then $\mathrm{SL}_2(\mathbb{C})$ acts on P_n via

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot f(x,y) = f(ax+cy, bx+dy)$$

For $n = 2$,

$$\begin{bmatrix} 1 & t \\ & 0 \end{bmatrix} \cdot x^2 = x^2; \quad \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix} \cdot xy = xy + tx^2; \quad \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix} \cdot y^2 = (tx+y)^2$$

and

Lemma 18.4. *Given $\beta \neq \pm\alpha$, let $\dim(V)_\beta = n+1$. Then there is an isomorphism of vector spaces $\varphi_\beta \longrightarrow P_n$ such that*

$$\varphi_\beta(u_\alpha(t)\cdot v) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \cdot \varphi_\beta(v); \quad \varphi_\beta(u_{-\alpha}(t)\cdot v) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \cdot \varphi_\beta(v)$$

Proof. (Sketch) without loss of generality $\beta - \alpha$ is not a root, else $v_\beta = v_{\beta-\alpha}$ and can do a replacement. Define

$$\varphi_\beta(e_{\beta+k\alpha}) = d_k x^k y^{n-k}$$

for some well chosen constants d_k (c.f. Carter, "Simple Groups of Lie Type," 16.2). □

Now, if $\beta \neq \pm\alpha$, we can define $\varphi_\alpha(x)(y) = x \cdot \varphi_\beta(y)$ for all $y \in V_\beta$, $x \in \mathrm{SL}_2(\mathbb{C})$. Note that by construction $\varphi_\alpha(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}) = u_\alpha(t) \in \mathrm{GL}(\mathfrak{g})$ and $\varphi_\alpha(\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}) = u_{-\alpha}(t) \in \mathrm{GL}(\mathfrak{g})$ and so
$im(\varphi_\alpha) = G_\alpha$.

□

The Weyl group acts on $\mathfrak{t}$ via $w \cdot h_{\alpha_i} = h_{w(\alpha_i)}$ for all i and extending linearly.

Exercise. Under this action, $w \cdot h_\alpha = h_{w(\alpha)}$ for all $w \in W, \alpha \in \Phi$.

Hint: Write $h_\alpha = \sum c_i h_{\alpha_i}$, Things are equal in $\mathfrak{t} \iff$ they are equal in every root, so is sufficient to prove the claim for $w = w_\beta$. Note that $\gamma(h_{w(\alpha)\beta}) = \langle \gamma, w_\alpha \check{\beta} \rangle$.

Proposition 18.5. *Let $n_\alpha = \varphi_\alpha(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}) \in G_{\mathrm{ad}}$. Then for all $\beta \in \Phi$,*

$$n_\alpha \cdot h_\beta = h_{w_\alpha(\beta)}; \quad n_\alpha \cdot e_\beta = \pm e_{w_\alpha(\beta)}$$

Proof. (Sketch): Action on m_α

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

so $n_\alpha \cdot e_\alpha = -e_{-\alpha}$, and similarly $n_\alpha \cdot e_{-\alpha} = -e_\alpha$, $n_\alpha \cdot h_\alpha = -h_\alpha = h_{-\alpha} = h_{w_\alpha(\alpha)}$. If $\beta \neq \pm\alpha$, then $h_\beta - \frac{1}{2}\langle \alpha, \check{\beta} \rangle h_\alpha \in \mathrm{Ker}(\alpha)$.

$$n_\alpha(h_\beta - \frac{1}{2}\langle \alpha, \check{\beta} \rangle h_\alpha) = h_\beta - \frac{1}{2}\langle \alpha, \check{\beta} \rangle h_\alpha$$

so

$$n_\alpha(h_\beta) + \frac{1}{2}\langle \alpha, \check{\beta} \rangle h_\alpha = h_\beta - \frac{1}{2}\langle \alpha, \check{\beta} \rangle h_\alpha$$

so

$$n_\alpha(h_\beta) = h_\beta - \langle \alpha, \check{\beta} \rangle h_\alpha$$

and it suffices to show that $\gamma(h_\beta - \langle \alpha, \check{\beta} \rangle h_\alpha) = \gamma(h_{w_\alpha(\beta)})$ for each $\gamma \in \Phi$.

The left hand side is $\langle \gamma, \check{\beta} \rangle - \langle \alpha, \check{\beta} \rangle \langle \gamma, \check{\alpha} \rangle$, and the right hand side is $\langle \gamma, w_\alpha \check{\beta} \rangle = \langle w_\alpha \gamma, \check{\beta} \rangle = \langle \gamma - \langle \gamma, \check{\alpha} \rangle, \check{\beta} \rangle$, so we are done. □

Generalising all of this:

1. Can replace ad with an arbitrary representation $\varphi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. Can define $u_\alpha(t) = \exp(\varphi(te_\alpha))$ for all t, α and the group generated by $\langle u_\alpha \mid \alpha \in \Phi \rangle$ is a subgroup of $\mathrm{GL}(V)$.

2. If careful, this can be done for any group.