# Introduction to Geometric Group Theory

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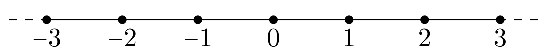
Based on lectures by Dr Ana Khukhro

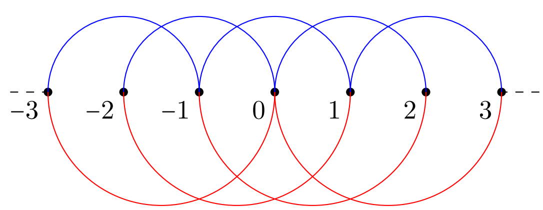
## 0 Introduction

The course will look at the following:

1. Free groups, which possess an important universal property. We'll use covering space theory to study subgroups of free groups.
2. Group presentations and constructions: a nice way to write down groups, and way of making new groups from old.
3. Cayley graphs, e.g. for $\mathbb{Z}$ viewed as $\langle 1 \rangle$ the Cayley graph is

   we view things as "close together" if they are separated by a generator of the group. So if instead we view $\mathbb{Z} = \langle 2, 3 \rangle$ we instead get the following Cayley graph, with the blue edges between the elements separated by 2 and the red edges those separated by 3:

   We will establish a notion of equivalence up to which these graphs are the same.
4. Viewing groups geometrically, and connections to group actions.
5. Geometric properties of groups, the geometry of free groups, and a "dictionary" between algebra and geometry.
6. Analytic properties of groups, amenability and the Banach-Tarski paradox.

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# 1 Free groups

Let $S$ be a set, called an *alphabet* and let $S^{-1} = \{s^{-1} : s \in S\}$ be the set of formal inverses of elements of $S$.

A *word* in the alphabet is a finite sequence of elements of $S \cup S^{-1}$, $s_1 s_2 s_3, \cdots s_n$, including the empty word. We call a word *reduced* if it does not contain occurences of the form $ss^{-1}$ or $s^{-1}s$. Given any word, we can get a reduced word by removing any such subwords, e.g. if $S = \{a, b, c\}$,

$$aa^{-1}bcb^{-1}bc^{-1}b \rightsquigarrow bcc^{-1}b \rightsquigarrow b^2$$

Adding or removing $ss^{-1}, s^{-1}s$ defines an equivalence relation on the set of words, and each equivalence class contains exactly one reduced word.

**Definition.** The *free group* on the set $S$, denoted by $F(S)$ is the set of reduced words in the alphabet $S$ with the operation of concatenation followed by reduction, that is,

$$(s_1 s_2 \cdots s_n)(t_1 t_2 \cdots t_n) = s_1 s_2 \cdots s_n t_1 t_2 \cdots t_n$$

followed by reduction if $s_n = t_1^{-1}$ or $t_1 = s_n^{-1}$.

Free groups satisfy an important universal property.

**Theorem 1.1.** *Given a free group $F(S)$ on a set $S$ with the inclusion $\iota : S \hookrightarrow F(S)$, whenever $G$ is a group with a function $\varphi : S \longrightarrow G$, there is a unique group homomorphism $\overline{\varphi} : F(S) \longrightarrow G$ such that the following diagram commutes:*

$$\begin{tikzcd} S \arrow[r, hook, "\iota"] \arrow[dr, "\varphi"'] & F(S) \arrow[d, dashed, "\overline{\varphi}"] \\ & G \end{tikzcd}$$

*So, we have a 1-1 correspondence*

$$\{\textit{homomorphisms } F(S) \longrightarrow G\} \leftrightarrow \{\textit{functions } S \longrightarrow G\}$$

*Proof.* Given a function $\varphi : S \longrightarrow G$, define the homomorphism $\overline{\varphi} : F(S) \longrightarrow G$ by

$$(s_{i_1}^{\alpha_1} s_{i_2}^{\alpha_2} \cdots s_{i_n}^{\alpha_n}) \mapsto \varphi(s_{i_1})^{\alpha_1} \varphi(s_{i_2})^{\alpha_2} \cdots \varphi(s_{i_n})^{\alpha_n}$$

This is a homomorphism because any occurence of $ss^{-1}$ or $s^{-1}s$ in a product of two reduced words will contribute $\varphi(s)^{-1}\varphi(s)$ or $\varphi(s)\varphi(s)^{-1}$ to the value of $\overline{\varphi}$, so we can reduce without changing $\overline{\varphi}$. $\square$

**Definition.** The cardinality of the set $S$, $|S|$ is the *rank* of $F(S)$, denoted by $\mathrm{rk}(F(S)$.

**Corollary 1.2.** *If $|S| = |T|$ then $F(S) \cong F(T)$.*

*Proof.* If $|S| = |T|$ then there is a bijection $\theta : S \longrightarrow T$. Consider the following diagram

$$\begin{tikzcd}
S \arrow[r, hook] \arrow[dr, "\iota \circ \theta"'] & F(S) \arrow[d, dashed, "\overline{\theta}"] \\
& F(T)
\end{tikzcd}$$

By Theorem 1.1 there is a homomorphism, $\overline{\theta}$ extending $\theta$. Similarly $\theta^{-1} : T \longrightarrow S$ gives a homomorphism $F(T) \longrightarrow F(S)$ namely $\overline{\theta^{-1}}$. Moreover, $\theta^{-1} \circ \theta : F(S) \longrightarrow F(S)$ is a homomorphism extending the identity map so by uniqueness is the identity. So $\theta, \theta^{-1}$ are isomorphisms. □

**Notation.** We write $F_n$ for the (isomorphism classes of) $F(S)$ with $|S| = n$.

**Remark.** It is in the exercises that if $F_n \cong F_m$ then $n = m$.

**Corollary 1.3.** *Every group is a quotient of a free group.*

*Proof.* Given a group $G$, it suffices to take $F(G)$. By Theorem 1.1 there is a homomorphism $\pi : F(G) \longrightarrow G$ extending the inclusion map which is surjective, so by the first isomorphism theorem $G$ is isomorphic to a quotient of $F(G)$. □

**Definition.** Let $G$ be a group and $A$ a subset of $G$. Denote by $\langle A \rangle$ the smallest subgroup of $G$ containing all elements of $A$, the intersection of all subgroups containing $A$. $\langle A \rangle$ has the following explicit description

$$\langle A \rangle = \{\alpha_1^{\epsilon_1} \alpha_2^{\epsilon_2} \cdots \alpha_n^{\epsilon_n} : n \in \mathbb{N}, \alpha_i \in A, \epsilon_i = \pm 1\}$$

It can be checked that this is indeed a subgroup.

**Definition.** We say that $G$ is *generated by* $A \subseteq G$ if $\langle A \rangle = G$. In this case, $A$ is a *generating set for* $G$. $G$ is *finitely generated* if there is a finite $A$ such that $\langle A \rangle = G$.

**Notation.** Given $\{a_1, \dots, a_n\} \subseteq G$ we write $\langle a_1, \dots, a_n \rangle$ for $\langle \{a_1, \dots, a_n\} \rangle$.

**Examples.**

1. $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$ can be generated by a single element. We call these groups *cyclic*.
2. $\mathbb{Z}^n$ can be generated by $n$ or more elements.
3. $F_2 = \langle a, b \rangle = \langle a, b \rangle = \langle a, b, ab^2 \rangle$.

**Remark.** Generating sets are not unique!

**Definition.** A group $F$ is *freely generated* by a set $S \subseteq F$ is for any group $G$ and any map $\varphi : F(S) \longrightarrow G$ there is a unique homomorphism $\overline{\varphi} : F \longrightarrow G$ extending $\varphi$, (the universal property with respect to $S$).

**Lemma 1.4.** *If $F$ is freely generated by $S$ then $F = \langle S \rangle$.*

*Proof.* (Exercise.) It is immediate from the definition that $\langle S\rangle$ is freely generated by $S$. Suppose $F$ is also freely generated by $S$.Let $\iota : S \longrightarrow \langle S\rangle$ be inclusion and let $\bar{\iota} : F \longrightarrow \langle S\rangle$ be the unique extension of $\iota$ that exists by the universal property. Both $S$ and $\langle S\rangle$ include into $F$, so consider the following diagram.

$$S \hookrightarrow F \xhookrightarrow{\bar{\iota}} \langle S\rangle \hookrightarrow F$$

Let $\phi : S \longrightarrow F$ be the composition of these maps. Then, $\mathrm{Id} : F \longrightarrow F$ extends $\phi$, so by uniqueness the composition of the two rightmost arrows must be Id. Therefore, $\bar{\iota}$ is injective, and since it extends $S \hookrightarrow \langle S\rangle$ it must be surjective, and hence it is an isomorphism. So, $F = \langle S\rangle$. $\square$

Let us now consider subgroups of free groups.

- Given $e \neq w \in F_n$, $\langle w\rangle \cong \mathbb{Z}$.
- Give $T \subseteq S$, $\langle T\rangle$ is a free subgroup of $F(S)$ of rank $|T|$.
- If $S = \langle\{a, b\}\rangle$ then the set $\{a^{-n}ba^n\}$ freely generates $F_\infty \leq F_2$.

**Remark.** A subgroup of a finitely generated group is not necessarily finitely generated. In fact, all subgroups of a free group are free. We will need some algebraic topology to show this.

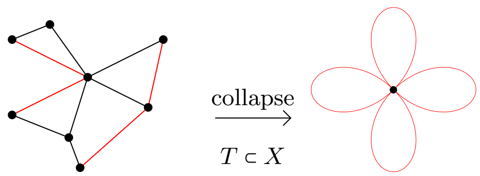
Let $X$ be a topological space. A *loop* based at a fixed basepoint $x_0 \in X$ is a continuous function $\alpha : [0,1] \longrightarrow X$ such that $\alpha(0) = \alpha(1) = x_0$. Two loops $\alpha, \beta$ are *homotopic* if we can continuously deform one to the other, i.e. there is a continuous $F : [0,1]^2 \longrightarrow X$ such that $F(t,0) = \alpha(t)$, $F(t,1) = \beta(t)$, and $F(0,u) = F(1,u) = x_0$ for all $t, u \in [0,1]$. This defines an equivalence relation on the set of loops based at $x_0$. Given two loops $\alpha, \beta$ we write $\alpha * \beta$ for the concatenation: the loop that does $\alpha$ at double speed then $\beta$ at double speed.

**Definition.** The set of homotopy classes of loops based at $x_0$ with concatenation is the *fundamental group* $\pi_1(X, x_0)$. For a path connected space, we can just write $\pi_1(X)$ since $\pi_1(X, x_0) = \pi_1(X, y_0)$ if $x_0, y_0$ are connected by a path.

**Examples.**

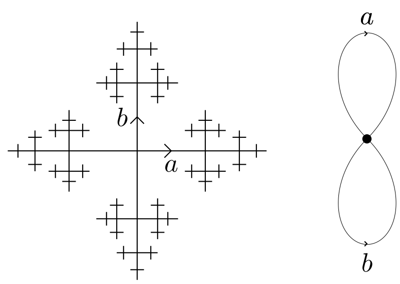
1. $\pi_1(\text{point}) = \{1\}$.
2. $\pi_1(S^1) = \mathbb{Z}$.
3. $\pi_1(S^1 \times [0,1]) = \mathbb{Z}$.
4. $\pi_1(S^1 \times \mathbb{R}) = \mathbb{Z}$.
5. $\pi_1(\text{torus}) = \mathbb{Z}^2$.
6. $\pi_1(\bigwedge_{i=1}^n S^1) = F_n$.
7. $\pi_1(\text{tree}) = \{1\}$.

**Fact.** If $Y \subseteq X$ is a nice topological space (e.g. a graph) with $Y$ simply connected and closed then collapsing $Y$ to a point does not alter $\pi_1(X)$.




So, since maximal trees always exist (with the axiom of choice if a graph is infinite) then $\pi_1(\text{graph})$ is the free group of rank $n$, where $n$ is the number of edges not in a maximal tree.

Now, recall the Galois correspondence between subgroups of $\pi_1(X)$ and covering spaces, i.e. coverings $\rho : (\overline{X}, x_0) \longrightarrow (X, x_0)$ correspond to the subgroups of $\pi_1(X, x_0)$. That is, for any $H \leq \pi_1(X, x_0) = F_n$ (in our case) there is a covering space $\overline{X}$ with $\pi_1(\overline{X}) \cong H$. Since a covering space of a graph is a graph, this gives that subgroups of free groups are free.




It turns out that given the rank of a (finite index) subgroup $H$ of $F_n$, given its index. The index is the degree of the covering map $\rho : \overline{X} \longrightarrow X$ corresponding to $H$, that is, the number of vertices in $\overline{X}$. There are $2n$ edges coming out of each vertex of $\overline{X}$ (since it is a cover of a bouquet of $n$ circles), so there are $[F_n : H] \cdot \frac{2n}{2}$ edges in $\overline{X}$. A tree on $[F_n : H]$ vertices has $[F_n : H] - 1$ edges, so the number of edges not in a maximal spanning tree is

$$[F_n : H] \cdot n - ([F_n : H] - 1) = (n - 1)[F_n : H] + 1.$$

So, we get the Neilson-Schreier formula, which says the following.

**Theorem 1.5.** *Ever subgroup $H < F$ of a free group $F$ is free and the rank satisfies*

$$\operatorname{rk}(H) - 1 = [F : H](\operatorname{rk}(F) - 1).$$

Recall that given two covering spaces $(p, \overline{X})$ and $(q, \widetilde{X})$

$$\begin{tikzcd} \overline{X} \arrow[dr, "p"] & & \widetilde{X} \arrow[dl, "q"'] \\ & X & \end{tikzcd}$$

we say that the two covers are *isomorphic as covering spaces* of $X$ if there is an $f : \overline{X} \longrightarrow \widetilde{X}$ such that $q \circ f = p$.

The group of *covering transformations* or *deck transformations* of a cover $\rho : \overline{X} \longrightarrow X$ is the group of isomorphisms $\overline{X} \longrightarrow \overline{X}$. The cover is *normal* if for any two lifts of the base point $x_0 \in X$ there is a covering transformation of $\overline{X}$ sending one of these lifts of the basepoint to the other. Recall that a covering space is normal if and only if the corresponding subgroup is normal in $\pi_1(X)$. Moreover, if the cover is normal with corresponding subgroup $N$, the group of covering transformations is isomorphic to $\pi_1(X)/N$.

**Example.** $F_3 \lhd F_2$ where the group of covering transformations is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

# 2 Group presentations and constructions

The *normal closure* of a subset $A \subseteq G$, denoted $\langle\!\langle A \rangle\!\rangle$ is the unique smallest normal subgroup of $G$ containing $A$. We write $\langle\!\langle a_1, \dots, a_n \rangle\!\rangle$ for $\langle\!\langle \{a_1, \dots, a_n\} \rangle\!\rangle$.

**Exercise.** $\langle\!\langle A \rangle\!\rangle$ is generated by the set $\{g^{-1}ag \mid g \in G, a \in A\}$.

Given a free group $F(S)$ and a subset $R \subseteq F(S)$ we write $\langle S \mid R \rangle$ for the group
$$\langle S \mid R \rangle = F(S)/\langle\!\langle R \rangle\!\rangle.$$
The elements of $S$ are called *generators* and the elements of $R$ are called *relators*.

**Definition.** A *presentation* of a group $G$ is an isomorphism of $G$ with a group of the form $\langle S \mid R \rangle$. $G$ is *finitely presented* if it admits a presentation $\langle S \mid R \rangle$ with $|S|, |R|$ finite.

**Examples.**

1. If $R = \varnothing$, then $\langle S \mid F \rangle \cong F(S)$.
2. $\langle a \mid a^n \rangle \cong \mathbb{Z}/n\mathbb{Z}$.
3. $\langle a, b \mid aba^{-1}b^{-1} \rangle \cong \mathbb{Z}^2$. To see this, $\mathbb{Z}^2 = \{(c^n, d^m) \mid n, m \in \mathbb{Z}\}$ with componentwise multiplication, and $\langle a, b \mid aba^{-1}b^{-1} \rangle \cong F(a,b)/\langle\!\langle aba^{-1}b^{-1} \rangle\!\rangle$. We can think of $\mathbb{Z}^2$ as a quotient of $F(a,b)$ via the homomorphism $\widetilde{\varphi} : F(a,b) \longrightarrow \mathbb{Z}^2$ which extends the map $\varphi : \{a, b\} \longrightarrow \mathbb{Z}^2$ sending $a \mapsto c$, $b \mapsto d$ by the universal property of a free group. So we just need to know that $\ker \widetilde{\varphi} = \langle\!\langle aba^{-1}b^{-1} \rangle\!\rangle$. The containment of the right hand side in the left is clear since $\mathbb{Z}^2$ is abelian. So, there is a surjection
$$F(a,b)/\langle\!\langle aba^{-1}b^{-1} \rangle\!\rangle \twoheadrightarrow F(a,b)/\ker(\widetilde{\varphi}) \cong \mathbb{Z}^2.$$
Moreover, the only two-generated abelian group that can surject onto $\mathbb{Z}^2$ is $\mathbb{Z}^2$.
4. More generally, if an abelian group is finitely generated then it is also finitely presented.

**Exercise.** This is also true for *nilpotent* groups. Recall that $G$ is nilpotent if the lower central series of $G$ terminates in finitely many steps, so there is an $n$ such that

$$G_0 = G \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = \{1\}, \quad G_{i+1} = [G_i, G] = \langle [h,g] \mid h \in G_i, g \in G \rangle.$$

5. $\langle a, b \mid a^3, b^2, a^{-2}b^{-1}ab \rangle \cong S_3$.
6. $\langle a, b \mid aba^{-1}b^{-2}, a^{-2}b^{-1}ab \rangle \cong S_3$.

**Remarks.** In general, it's difficult to recognise which group a particular presentation represents. Indeed, there does not exist an algorithm that, upon input of a presentation, can determine whether the corresponding group is trivial. This is the *word problem* introduced by Dehn in the early twentieth century. The classes for which the problem is solved have certain geometric properties.

There is only a countable number of isomorphism classes of finitely presented groups (but uncountably many isomorphism classes of 2-generated groups - see

In fact, the notation of "finite presentability" of a group makes sense without fixing a specific surjection onto a free group.

**Proposition 2.1.** *Given a (not necessarily finite) presentation of a finitely presented group,*

$$G \cong \langle (s_j)_{j \in J} \mid (r_i)_{i \in I} \rangle,$$

*there exists a finite subset $J_0 \subset J$ and a finite set of elements $R_0$ in $F(J_0)$ such that*

$$G \cong \langle (s_j)_{j \in J_0} \mid R_0 \rangle.$$

Our goal for now will be to show that a finite index subgroup of a finitely presented (respectively, finitely generated) group is also finitely presented (respectively, finitely generated).

**Definition.** Let $F(S)$ be a free group and $H < F(S)$ a subgroup. A (right) *Schreier transversal* for $H$ in $F(S)$ is a set $J$ of reduced words such that each right coset of $H$ in $F(S)$ contains a unique element of $J$ (a *representative* for this class) and all initial segments of those words also lie in $J$.

For $g \in F(S)$ we denote by $\overline{g}$ the element of $J$ such that $Hg = H\overline{g}$.

**Theorem 2.2.** *For any $H < F(S)$, there exists a Schreier transversal $J$. Moreover, $H$ is freely generated by the set*

$$\{ts(\overline{ts})^{-1} \mid t \in J, s \in S, ts(\overline{ts})^{-1} \neq 1\}.$$

*Proof.* Let $X$ be a bouquet of $|S|$ circles. Then $\pi_1(X) \cong F(S)$. Let $\overline{X}$ be the cover corresponding to $H$, so $\pi_1(\overline{X}) \cong H$.

**Exercise.** The vertices of $\overline{X}$ correspond to the cosets of $H$ in $F(S)$.

Choosing a path labelled by elements of $S$ according to the covering to a given vertex from the basepoint gives a coset representative.

Thus, picking a maximal tree $T \subset \overline{X}$, the unique paths from basepoint to vertices give us coset representatives where the initial segments are also paths in $T$, and hence are also specified as coset representatives.

Moreover, since $H \cong \pi_1(\overline{X})$ and is freely generated by the set of loops with exactly one edge not in $T$. Therefore the generating set is exactly

$$\{(ts)(\overline{ts})^{-1} \mid t \in T, s \in S, ts(\overline{ts})^{-1} \neq 1\}.$$

□

**Remark.** The proof actually shows that the set of Schreier transversals for $H$ in $F(S)$ is in bijection with the set of maximal trees in the cover corresponding to $H$.

**Notation.** Write $\gamma(t,s) = ts(\overline{ts})^{-1}$.

Given $H < F(S)$, let $B$ denote the free generating set of $H$ given by Theorem 2.2. Given $h \in H$ written as a word $s_1 \dots s_n$ with $s_i \in S \cup S^{-1}$, we can rewrite it as a word in the generating set $B$ as follows:

$$h = \gamma(1, s_1) \cdot \gamma(\overline{s_1}, s_2) \cdot \ldots \cdot \gamma(\overline{s_1 \dots s_{n-1}}, s_n).$$

Since $\gamma(t, s^{-1}) = \gamma(\overline{ts^{-1}}, s)^{-1}$ we can use this process (called the *Reidemeister-Schreier rewriting process*) to rewrite $h$ as a word in $B$.

**Theorem 2.3.** *Let $G$ be a group with presentation $\langle S \mid R\rangle$ and let $\varphi : F(S) \twoheadrightarrow G$ be the surjective homomorphism corresponding to this presentation. Let $G_1 < G$ and let $H$ be the subgroup of $F(S)$ containing* $\mathrm{Ker}(\varphi)$ *such that $\varphi(H) = G_1$. Then $G_1$ has the presentation*

$$\langle \gamma(t, st) : t \in J, s \in S, \gamma(t,s) \neq 1 \mid trt^{-1} : t \in J, r \in R\rangle.$$

**Notation.** Write $\langle\!\langle R\rangle\!\rangle^G$ for the normal closure of $R$ in $G$.

*Proof.* We have $G_1 \cong H/\langle\!\langle R\rangle\!\rangle^{F(S)}$, and $H$ has free generating set

$$\{\gamma(t,s) : t \in J, s \in S, \gamma(t,s) \neq 1\}.$$

The subgroup $\langle\!\langle R\rangle\!\rangle^{F(S)}$ is generated by

$$\{grg^{-1} : g \in F(S), r \in R\},$$

and writing each $g$ as $h_g\overline{g}$ for $h_g \in H$ we get that

$$\begin{aligned} grg^{-1} &= (h_g\overline{g}r(h_g\overline{g})^{-1} \\ &= h_g(\overline{g}r\overline{g}^{-1})h_g^{-1} \end{aligned}$$

so

$$\langle\!\langle R\rangle\!\rangle^{F(S)} = \langle\!\langle trt^{-1} : t \in J, r \in R\rangle\!\rangle^{H}$$

so

$$G_1 = \langle \gamma(t, st) : t \in J, s \in S, \gamma(t,s) \neq 1 \mid trt^{-1} : t \in J, r \in R\rangle.$$

□

**Corollary 2.4.** *Any subgroup of finite index in a finitely presented (respectively, finitely generated) group is itself finitely presented (respectively, finitely generated).*

We now turn to ways of creating new finitely generated or finitely presented groups out of old ones. The first construction we will see is the free product.

**Definition.** Given two groups $A, B$ such that $A \cap B = \{1\}$ a *normal form* is an expression of the form $g_1 \dots g_n$ for some $n \geq 0$ such that $g_i \in A \cup B \backslash \{1\}$ and consecutive $g_i, g_{i+1}$ lie in a different group (out of $A$ and $B$). The *length* of the normal form is $n$.

We define a multiplication of normal forms as follows (by induction on the length).

- $(g_1 \dots g_n) \cdot 1 = 1 \cdot (g_1 \dots g_n) = g_1 \dots g_n$.
- For $n, m \geq 1$, if $g = g_1 \dots g_n$ and $h = h_1 \dots h_m$ set

$$gh = \begin{cases} g_1 \dots g_n h_1 \dots h_m & \text{if } g_n, h_1 \text{ in different groups} \\ g_1 \dots g_{n-1} k h_2 \dots h_m & \text{if } g_n, h_1 \text{ in same group with } g_n h_1 = k \neq 1 \\ g_1 \dots g_{n-1} h_2 \dots h_m & \text{if } g_n, h_1 \text{ in same group with } g_n h_1 = 1. \end{cases}$$

The set of normal forms with this multiplication form a group $A * B$ called the *free product* of $A$ and $B$.

**Remarks.**

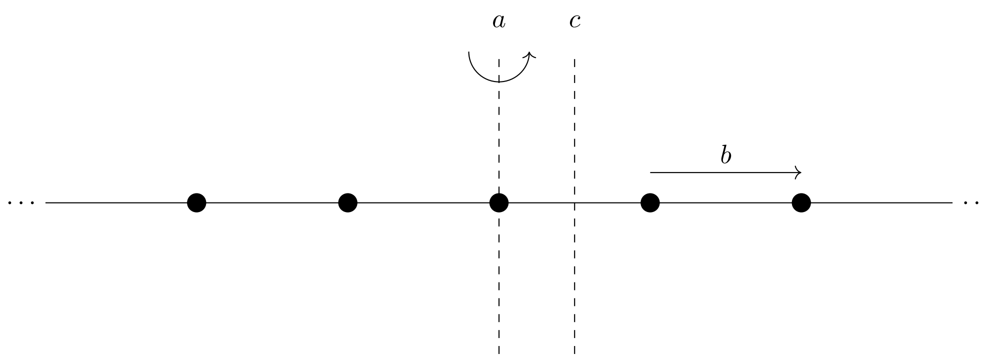
1. The groups $A, B$ are naturally embedded into $A * B$.
2. If $A, B < G$ are such that for any $(g \neq 1) \in G$ can be represented in a unique way as a product $g = g_1 \dots g_n$ with $g_i \in A \cup B \backslash \{1\}$ and consecutive $g_i, g_{i+1}$ are not in the same group, then $G \cong A * B$.

**Theorem 2.5.** *If $A = \langle S_A \mid R_A\rangle$ and $B = \langle S_B \mid R_B\rangle$ and $S_A \cap S_B = \varnothing$ then $A * B = \langle S_A \cup S_B \mid R_A \cup R_B\rangle$.*

*Proof.* Let $\varphi : F(S_A) \twoheadrightarrow A$ be the homomorphism with kernel $\langle\!\langle R_A\rangle\!\rangle^{F(S_A)}$, and let $\psi$ be analogous for $B$. Define $\theta : F(S_A \cup S_B) \longrightarrow A * B$ to be the homomorphism coinciding with $\varphi$ on $S_A$ and with $\psi$ on $S_B$. We need to show that $\mathrm{Ker}(\theta) = \langle\!\langle R\rangle\!\rangle^{F(S_A \cup S_B)}$.

The containment $\supseteq$ is clear. For the other containment, take $g = g_1 \dots g_n \in \mathrm{Ker}(\theta)$ with $g_i \in F(S_A) \cup F(S_B) \backslash \{1\}$ and consecutive $g_i, g_{i+1}$ not in the same group. Since $g \in \mathrm{Ker}(\theta)$ we have $\theta(g) = \theta(g_1) \dots \theta(g_n) = 1$, which is in $A \nleq B$. So, there must be some $i$ such that $\theta(g_i) = 1$. So $g_i \in \langle\!\langle R_A\rangle\!\rangle^{F(S_A)}$ or $\langle\!\langle R_B\rangle\!\rangle^{F(S_B)}$. We then have $\theta(g_1 \dots g_{i-1} g_{i+1} \dots g_n) = 1$ so we can reduce again, and continuing this process iteratively we have that $g_1 \dots g_n \in \langle\!\langle R_A \cup R_B\rangle\!\rangle^{F(S_A \cup S_B)}$. □

**Example.** $D_\infty \cong \langle a, b \mid a^2 = 1, a^{-1}ba = b^{-1} \rangle = \langle b \rangle \rtimes \langle a \rangle$ (which is also isomorphic to $\cong \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$).




Thinking of $D_\infty$ as the group of graph automorphisms of $C_\infty$, it can be generated by reflections in the two dashed lines above, and one can check that $(ca)^n, (ca)^n c, a(ca)^n c, a(ca)^n$ give different actions. Thus

$$D_\infty \cong \langle a, c \mid a^2 = c^2 = 1 \rangle = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}.$$

In fact, this is the only free product of two nontrivial groups not containing a nonabelian free group (i.e. $F_n$ for $n \geq 2$).

Recall that if a group $G$ acts on a set $X$, the action is *transitive* if for all $x, y \in X$ there is a $g \in G$ such that $g \cdot x = y$, *faithful* if $\bigcap_X \mathrm{Stab}_G(x) = \{1\}$ and *free* if there is an $x \in X$ such that $\mathrm{Stab}_G(x) = \{1\}$.

**Theorem 2.6** (Ping-pong Lemma)**.** *Let $G$ act on $X$ and let $H_1, H_2$ be subgroups of $G$ such that $|H_1| \geq 3$, $|H_2| \geq 2$, and $H = \langle H_1, H_2 \rangle$. Suppose there are nonempty subsets $X_1, X_2 \subset X$ with $X_2 \notin X_1$ such that:*

*(i) for all $h \in H_1 \backslash \{1\}$, $h(X_1) \subseteq X_2$, and*

*(ii) for all $h \in H_2 \backslash \{1\}$, $h(X_1) \subseteq X_1$.*

*Then $H = H_1 * H_2$.*

*Proof.* Let $w$ be a nonempty reduced word in the alphabet $H_1 \backslash \{1\} \sqcup H_2 \backslash \{1\}$. We need to show that $w \neq 1$.

If $w = a_1 b_1 a_2 b_2 \ldots a_k$with $a_i \in H_1 \backslash \{1\}, b_i \in H_2 \backslash \{1\}$ then

$$\begin{aligned} w(X_2) &= a_1 b_1 a_2 b_2 \ldots a_k(X_2) \\ &\subseteq a_1 b_1 a_2 b_2 \ldots b_{k-1}(X_1) \\ &\subseteq \ldots \\ &\subseteq a_1(X_2) \\ &\subseteq X_1. \end{aligned}$$

Since $X_2 \notin X_1$, $w \neq e$ in $G$ because it acts nontrivially on some $x_2 \in X_2$.

- If $w = b_1 \dots a_k b_k$ then apply above to $awa^{-1}$ with $a \in H_1 \backslash \{1\}$.
- If $w = a_1 b_1 \dots a_k b_k$ take $a \in H_1 \backslash \{1, a_1^{-1}\}$ and consider $awa^{-1}$.
- If $w = b_1 a_1 \dots a_k$, take $a \in H_1 \backslash \{1, a_k\}$ and repeat with $awa^{-1}$.

□

**Example.** We show $F_2 < \mathrm{SL}_2(\mathbb{Z})$ is a subgroup of finite index.

Let $\mathrm{SL}_2(\mathbb{Z})$ act on $\mathbb{Z}^2$ by matrix multiplication of vectors. Let

$$H_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 2n & 1 \end{bmatrix} : n \in \mathbb{Z} \right\} = \left\langle \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right\rangle \cong \mathbb{Z}$$

and let

$$H_2 = \left\{ \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\} = \left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right\rangle \cong \mathbb{Z}.$$

Set

$$X_1 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid |x| < |y| \right\} \text{ and } X_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid |x| > |y| \right\}.$$

Given $\begin{bmatrix} x \\ y \end{bmatrix} \in X_2$, we know $x \neq 0$ and

$$\begin{bmatrix} 1 & 0 \\ 2n & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2nx + y \end{bmatrix}.$$

Now, $|2nx + y| \geq |2n||x| - |y| \geq |2x| - |y| > |x|$. Similarly for $H_2$, and we get that $\langle H_1, H_2 \rangle \cong H_1 * H_2$ by the Ping-pong Lemma.

**Exercise.** Check that this subgroup is finite index.

## 2.1 More group constructions

1. Amalgamated free products: let $A \leq G$, $B \leq H$ such that $\varphi : A \longrightarrow B$ is an isomorphism. The free product of $G, H$ with amalgamation of $A, B$ (via $\varphi$) is
$$G *_A H = G * H / \langle\langle \varphi(a)^{-1} a \rangle\rangle^{G*H}.$$
**Example.** $\mathrm{SL}_2(\mathbb{Z}) = \mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$.

   We can define normal forms, $G, H$ embed as subgroups and there arise naturally in topology.

2. HNN extensions: let $A, B \leq G$ and let $\varphi : A \longrightarrow B$ be an isomorphism. The HNN extension of $G$ (with respect to $A, B, \varphi$) is
$$G *_\varphi = G * \langle t \rangle / \langle\langle t^{-1} a t \varphi(a)^{-1} : a \in A \rangle\rangle.$$
This makes $A, B$ isomorphic via conjugation in a bigger group. Again, we have normal forms. These arise e.g. if we take fundamental group of a surface bundle over $S^1$.

3. Semidirect products: $G$ is the semidirect product of $N$ by $H$ if $N \triangleleft G$, $H \le G$, $N \cap H = \{1\}$, and $NH = G$.

   Equivalently, there is a $\varphi : G \twoheadrightarrow H$ such that $\mathrm{Ker}(\varphi) = N$, i.e.

$$\{1\} \longrightarrow N \hookrightarrow G \longrightarrow H \longrightarrow \{1\}$$

   is exact. Suppose given a map $\alpha : H \longrightarrow \mathrm{Aut}(N)$. We can construct $G = N \ltimes_\alpha H$ by setting $G = N \times H$ as a set and defining a multiplication

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1 \alpha(h_1)(n_2), h_1 h_2).$$

   The subgroups $N \times \{1\}$ and $\{1\} \times H$ satisfy the above conditions: we can get this from the first definition by defining $\alpha : H \longrightarrow \mathrm{Aut}(N)$ to be $\alpha(h)(n) = h^{-1}nh$.

   **Examples.**
   - $D_{2n} \cong \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, where, if $b, a$ generate the subgroups respectively, $\langle a \rangle$ acts on $\langle b \rangle$ via $a^{-1}ba = b^{-1}$.
   - The fundamental group of the Klein bottle is $\langle\langle a, b \mid a^{-1}ba = b^{-1} \rangle\rangle$.
   - $N \times H$ has the trivial action.

   More generally, a *group extension* is a group $G$ given by data

$$\{1\} \longrightarrow N \hookrightarrow G \xrightarrow{\pi} H \longrightarrow \{1\}.$$

   The extension is *split* if there is a $\psi : H \longrightarrow G$ such that $\pi \circ \psi = \mathrm{Id}$ on $H$, and then $G = N \rtimes H$.

   **Exercise.** Any extension by a free group (i.e. where $H$ is free) splits.

   **Examples.**
   - The extension $2\mathbb{Z} \hookrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$ is not split.
   - The extension $\mathrm{Ker}(\varphi) \hookrightarrow F_n \xrightarrow{\varphi} G$ is not split if $G$ is not free.

4. Wreath Products: the wreath product $G \wr H$ of $G$ by $H$ is

$$\bigoplus_H G \rtimes H$$

   where, thinking of $\bigoplus_H G$ as the set of finitely supported functions $H \longrightarrow G$, the action of $H$ is given by

$$h(f)(h_1) = f(h^{-1}h_1).$$

   **Example.** The "Lamplighter group" is $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z} = \bigoplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}$. The elements are of the form $(\{a_i\}_{i\in\mathbb{Z}}, n)$ where $a_i \in \{0,1\}$ with finitely many nonzero $a_i$, and $n \in \mathbb{Z}$. This group is not finitely presented.

**Theorem 2.7** (Kaloujnine-Krasner, 1951)**.** *If $D, Q$ are groups with $Q$ finite then $D \wr Q$ contains an isomorphic copy of any extension of $D$ by $Q$.*

*Proof.* If $G$ is an extension of $D$ by $Q$, let $\pi : G \twoheadrightarrow Q$ with $\mathrm{Ker}(\pi) = D$. Choose a transversal for $D$ in $G$, and write it as a map $T : Q \longrightarrow G$. For $a \in G$, define $f_a : Q \longrightarrow D$ by $f_a(X) = T(X)^{-1} \cdot a \cdot T(\pi(a^{-1})x)$. Then $f_a \in \bigoplus_Q D$. If $a, b \in G$ then

$$\begin{aligned}(f_a(x)) \cdot (\pi(a))(f_b(x)) &= f_a(x) f_b(\pi(a^{-1}x) \\ &= T(x)^{-1} a T(\pi(a^{-1}x)) \cdot T(\pi(a^{-1}x))^{-1} b T(\pi(b^{-1}\pi(a^{-1})x)) \\ &= T(x)^{-1} ab T(\pi(ab)^{-1}x) \\ &= f_{ab}(x).\end{aligned}$$

Define $\varphi : G \longrightarrow D \wr Q$ by $\varphi(a) = (f_a, \pi(a))$. Then,

$$\begin{aligned}\varphi(a)\varphi(b) &= (f_a, \pi(a)) \cdot (f_b, \pi(b)) \\ &= (f_a \pi(a) f(b), \pi(ab)) \\ &= (f_{ab}, \pi(ab)).\end{aligned}$$

So, $\varphi$ is a homomorphism. To see that it is injective , let $a \in \mathrm{Ker}(\varphi)$, then $\pi(a) = 1$ and $f_a(x) = 1$ for all $x \in Q$, so $a \in D$ and

$$f_a(x) = T(x)^{-1} a T(\pi(a^{-1})x) = 1.$$

So, $T(x)^{-1} a T(x) = 1$, so $a = 1$. □

**Remark.** This can be extended to infinite groups $G, H$ by using $\prod_H G \rtimes H$.

# 3 Cayley Graphs

We now restrict to finitely generated groups.

**Definition.** Let $G$ be a finitely generated group and $S \subseteq G$ a subset. The *Cayley graph* of $G$ with respect to $S$, denoted by $\mathrm{Cay}(G, S)$ has the vertex set

$$V(\mathrm{Cay}(G, S)) = G,$$

and the edge set

$$E(\mathrm{Cay}(G, S)) = \{(g, gs) : g \in G, s \in S\}.$$

**Example.** Let $G = \mathbb{Z}^2$, and $S = \{(0,1), (1,0)\}$. Then we get the 2-dimensional grid. If instead we take $S = \{(1,0)\}$ we get all of the lines parallel to the $x$-axis in the 2-dimensional grid.

The Cayley graph has the following properties.

1. It is $2|S|$-regular.
2. $\mathrm{Cay}(G,S)$ is connected if and only if $\langle S\rangle = G$.
3. Relators in elements of $S$ give rise to cycles in the Cayley graph, and vice versa.
4. In the case $\langle S\rangle = G$, paths from 1 to a vertex give words in $S$ representing elements of $G$.
5. $\mathrm{Cay}(G,S)$ allows us to view $G$ as a metric space by setting

$$d_S(g,h) = \text{length of shortest path from g to h.}$$

So the word length $|g| = d_S(1,g)$. Note that $d_S(g,h) = |g^{-1}h|$. $G$ acts on $\mathrm{Cay}(G,S)$ by isometries via left-multiplication:

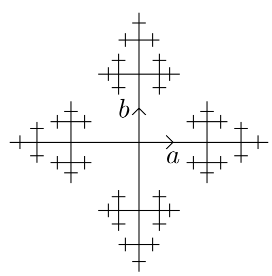
$$d_S(hg_1,hg_2) = d_s(g_1,g_2).$$

**Remarks.**

- A Cayley graph does not uniquely define a group: $\mathrm{Cay}(S_6,S_6) = \mathrm{Cay}(C_6,C_6) = K_6$.
- We refer to $G$ and $\mathrm{Cay}(G,S)$ interchangeably.

**Theorem 3.1.** $\mathrm{Cay}(F(S),S)$ *is the* $2|S|$*-regular tree.*

**Example.** For $F_2$ we get the universal cover of the wedge of two circles.




More generally, if $\overline{X}$ is a covering space of a bouquet of $|S|$ circles corresponding to $N \lhd F(S)$, then $\overline{X}$ is exactly the Cayley graph $\mathrm{Cay}(F(S)/N, \pi(S))$, where $\pi : F(S) \longrightarrow F(S)/N$.

If $N = \pi_1(\overline{X})$, we get $|F(S)/N|$ copies of the max tree. $F(S) = \pi_1(X)$ and contracting $T$ gives $\mathrm{Cay}(F(S)/N, \pi(S))$.

**Definition.** Let $(X,d_X)$ and $(Y,d_Y)$ be metric spaces. A map $f : X \longrightarrow Y$ is a *quasi-isometric embedding* if there is a $\lambda > 0$ and $c > 0$ such that

$$\frac{1}{\lambda}d_X(x_1,x_2) - c \le d_Y(f(x_1,f(x_2)) \le \lambda d_X(x_1,x_2) + c.$$

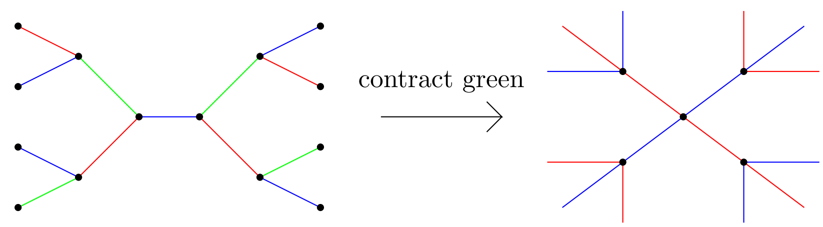
If additionally there is a $D \ge 0$ such that for all $y \in Y$ there is an $x \in X$ such that $d(y,f(x)) \le D$ then $f$ is a *quasi-isometry* and we write $X \simeq_{\mathrm{QI}} Y$. We also say $f$ is a $(\lambda,c,D)$-isometry.

Roughly speaking, quasi-isometry forgets local structure and sees the global structure.

**Proposition 3.2.** *Quasi-isometry is an equivalence relation.*

**Examples.**

1. Any nonempty bounded metric space is quasi-isometric to a point.
2. $\mathbb{R} \times [0,1] \simeq_{\mathrm{QI}} \mathbb{R}$.
3. Any Cayley graph of $\mathbb{Z}$ is quasi-isometric to $\mathbb{R}$.
4. Any Cayley graph of $\mathbb{Z}^d$ is quasi-isometric to $\mathbb{R}^d$ (so "quasi-isometry ignores holes").
5. The 3-regular tree is quasi-isometric to $T_4$ (so "quasi-isometry ignores degree"). To see this, colour the 3-regular tree's edges using three colours, one at each edge and collapse all the vertices of a fixed colour to a point:




Generally, for $n, m \geq 3$, $T_n \simeq_{\mathrm{QI}} T_m$.

**Non-examples.** To show spaces are not quasi-isometric, we often need to find properties that are invariant under quasi-isometry.

1. Boundedness is an invariant under quasi-isometry so $\mathbb{R}$ is not quasi-isometric to $\{*\}$.
2. $[0, \infty)$ is not quasi-isometric to $\mathbb{R}$.

*Proof.* Suppose $\varphi : \mathbb{R} \longrightarrow [0,\infty)$ is a quasi-isometry. Then as $t \longrightarrow \infty$, $\varphi(t) \longrightarrow \infty$, $\varphi(-t) \longrightarrow \infty$. For any $x$ sufficiently large, let

$$M_x = \max\{n \in \mathbb{Z} : \varphi(n) < x\} \text{ and } N_x = \min\{n \in \mathbb{Z} : \varphi(n) \leq x\}.$$

Then we have $\varphi(M_x) < x \leq \varphi(M_x + 1)$ and $\varphi(N_x) < x \leq \varphi(N_x - 1)$. Since

$$d_{\mathbb{R}}(M_x, M_x + 1) = 1 = d_{\mathbb{R}}(N_x, N_x - 1)$$

we get that

$$\begin{aligned} d_{[0,\infty)}(\varphi(M_x), \varphi(N_x)) &\leq d_{[0,\infty)}(\varphi(M_x), x) + d_{[0,\infty)}(\varphi(N_x), x) \\ &\leq d_{[0,\infty)}(\varphi(M_x), \varphi(M_x+1)) + d_{[0,\infty)}(\varphi(N_x), \varphi(N_x - 1)) \\ &\leq \lambda d_{\mathbb{R}}(M_x, M_x+1) + \lambda d_{\mathbb{R}}(N_x, N_x - 1) + 2c \\ &= 2(\lambda + c). \end{aligned}$$

This bound is independent of $x$, but $d_{\mathbb{R}}(M_x, N_x) \longrightarrow \infty$ as $x \longrightarrow \infty$, a contradiction. □

3. Quasi-isometry does see dimension:

   **Exercise.** $\mathbb{R}^2$ is not quasi-isometric to $\mathbb{R}$.

4. The contraction trick does require that our regular tree is at least 3-regular:

   **Exercise.** $T_3$ is not quasi-isometric to $\mathbb{R}$.

**Proposition 3.3.** *Let $G$ be finitely generated and let $S, S'$ be two finite generating sets for $G$. Then $(G, d_S) \simeq_{QI} (G, d_{S'})$.*

*Proof.* Let $\varphi : (G, d_s) \longrightarrow (G, d_{S'})$ be the identity map. Let $\lambda = \max\{|a|_{S'} : a \in S\}$, so that $d_{S'}(\varphi(g), \varphi(h)) \leq \lambda d_s(g, h)$ for all $g, h \in G$. Similarly, if $\lambda' = \max\{|b|_S : b \in S'\}$ we get $d_S(g, h) \leq \lambda' d_{S'}(\varphi(g), \varphi(h))$ for all $g, h \in G$. □

Recall that a metric space is *proper* if all closed balls are compact, and *geodesic* if for any two points there is a path between them realizing the distance between them.

**Definition.** An action of a group $G$ on a metric space $X$ is *proper* if for any compact subset $K \subset X$,

$$|\{g \in G \mid gK \cap K \neq \varnothing\}| < \infty.$$

Note that if $G$ acts on $X$ properly then $X/G(X)$ is Hausdorff and locally compact.

**Theorem 3.4** (Švarc-Milner Lemma)**.** *Let $X$ be a geodesic metric space and let $G$ act on $X$ properly by isometries with compact quotient (or cocompactly). Then $G$ is finitely generated and picking $x_0 \in X$ defines a quasi-isometry $\varphi : G \longrightarrow X$ sending $g \mapsto gx_0$.*

*Proof.* First we define our candidate for a finite generating set. Since the action is cocompact, there is a closed ball $\overline{B} = \overline{B}(x_0, D)$ such that $G\overline{B} = X$. Define $S$ to be

$$S = \{s \in G \mid s \neq 1, s\overline{B} \cap \overline{B} \neq \varnothing\}.$$

Since the action is proper, $S$ is finite.

Next we check what happens outside of $S$. For $A, B \subset X$, define

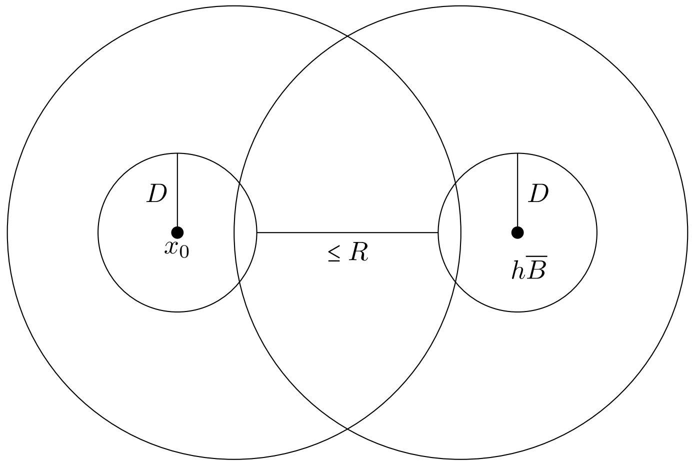
$$d(A, B) = \inf\{d_X(a, b) \mid a \in A, b \in B\}.$$

Consider $\inf\{d(\overline{B}, g\overline{B}) \mid g \in G\backslash(S \cup \{1\})\}$.Pick some $g \in G\backslash(S \cup \{1\})$, and let $d(\overline{B}, g\overline{B}) = R > 0$. We know this distance is positive because $g$ does not lie in $S$.

Now define

$$\begin{aligned} H &= \{h \in G \mid d(\overline{B}, h\overline{B}) \leq R\}. \\ &\subset \{g \in G \mid g\overline{B}(x_0, D + R) \cap \overline{B}(x_0, D + R) \neq \varnothing\} \end{aligned}$$

This is a finite set since the action is proper, so $H$ is finite.




So, $\inf\{d(\overline{B}, g\overline{B} \mid g \in G\backslash(S \cup \{1\})\} = \inf\{d(\overline{B}, g\overline{B} \mid g \in H\backslash(S \cup \{1\})\}$ is achieved, say at $h' \in H\backslash(S \cup \{1\})$. Set $d(\overline{B}, h\overline{B}) = 2d$. Now, if $d(\overline{B}, g\overline{B}) < 2d$ we know that $g \in S \cup \{1\}$.

Next we show $G$ is finitely generated by $S$. Fix $g \in G$ and take a geodesic $[x_0, gx_0]$. Set

$$k = \left\lfloor \frac{d_x(x_0, gx_0)}{d} \right\rfloor.$$


Take a sequence of points on the geodesic, $y_0 = x_0, y_1, y_2, \ldots, y_{k+1} = g_{x_0}$ such that $d(y_i, y_{i+1}) \leq d$ for all $i$. Take a corresponding sequence $h_i \in G$ such that for all $i$, $y_i \in h_i\overline{B}$, setting $h_0 = 1, h_{k+1} = g$. We have

$$d(h_i\overline{B}) \leq d_X(y_i, y_{i+1}) \leq d.$$

We also have $d(h_i\overline{B}, h_{i+1}\overline{B}) = d(\overline{B}, h_i^{-1}h_{i+1}\overline{B})$, so $h_i^{-1}h_{i+1} \in S \cup \{1\}$. Therefore $h_{i+1} = h_i s$. Applying this inductively, $g = h_{k+1} = s_0 \ldots s_k \in \langle S \rangle^G$.

Finally, we show that $G \simeq_{\mathrm{QI}} X$. All word metrics on $G$ are quasi-isometric so we can consider $d_S$ with $S$ as above. The $2D$-neighbourhood of the image of the map $\varphi_{x_0} : g \mapsto gx_0$ is $X$ (as $G(\overline{B}(x_0, D)) = X$), so we just need to check $\varphi_{x_0}$ is a quasi-isometric embedding.

By the above, $|g|_S = d_S(1, g) \leq k + 1 \leq \frac{d_X(x_0, gx_0)}{d} + 1$. Now, if $|g|_S = m$, and $t_1 \ldots t_m = g$ with $t_i \in S$ then

$$\begin{aligned}
d_X(x_0, gx_0) &= d(x_0, t_1t_2 \ldots t_m x_0) \\
&\leq d_X(x_0, t_1x_0) + d_X(t_1x_0, t_2t_1x_0) + \ldots + d_X(t_1 \ldots t_{n-1}x_0, t_1 \ldots t_m x_0) \\
&= \sum_{i=1}^{m} d_X(x_0, t_ix_0) \\
&\leq 2Dm \\
&= 2D|g|_S
\end{aligned}$$

Then we have

$$d \cdot d_S(1,g) - d \leq d_X(x_0, gx_0) \leq 2D d_S(1,g).$$

By the left-equivariance of $d_S$ and $d_X$ under the action of $G$, this is enough to show that $\varphi_{x_0}$ is a quasi-isometry. □

**Corollary 3.5.** *Let $M$ be a compact connected Riemannian manifold, and let $\widetilde{M}$ be its universal cover with the pullback Riemannian metric. Then the fundamental group of $M$ acts on $\widetilde{M}$ isometrically, and we get that $\pi_1(M)$ is finitely generated and quasi-isometric to $\widetilde{M}$.*

**Corollary 3.6.** *Let $G$ be a connected real Lie group and let $\Gamma$ be a connected lattice in $G$. Then $\Gamma$ is finitely generated and $G \simeq_{QI} \Gamma$ (where $G$ is viewed with some left-invariant Riemannian metric).*

**Definition.** We say groups $G, H$ are *commensurable* if there are finite index subgroups $K_1 \leq G, K_2 \leq H$ such that $K_1 \cong K_2$.

**Corollary 3.7.** *Let $G$ be a finitely generated group.*

1. *If $H$ is a finite index subgroup of $G$, $G \simeq_{QI} H$.*
2. *If $H$ is a finite normal subgroup of $G$, $G \simeq_{QI} G/H$.*
3. *If $G$ and $H$ are commensurable then $G \simeq_{QI} H$.*

*Proof.* For 1, let $H$ act on $\mathrm{Cay}(G)$, for 2, let $G$ act on $\mathrm{Cay}(G/N)$ and, for 3, use 1. □

**Corollary 3.8.** *All finitely generated nonabelian free groups are quasi-isometric.*

*Proof.* There is a finite index inclusion of $F_n$ into $F_2$ for all $n \geq 2$. □

The converse to corollary 3.7 is false: there are groups which are not commensurable but are quasi-isometric.

**Example.** Take $G = \mathbb{Z}/4\mathbb{Z} \wr \mathbb{Z}$ and $H = (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$. To see that these groups are not commensurable, the only elements of finite order in $H$ are of order 2, which also holds in any finite index subgroup of $H$. If $K_2 \leq G$ is a finite index subgroup of $G$ then $K_2$ necessarily contains an element of order 4, since the index

$$\left[\bigoplus_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z} : \mathbb{Z}/4\mathbb{Z} \cap K_2\right]$$

is finite but the subgroup generated by elements of order 2 in $\bigoplus_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$ has infinite index in $\bigoplus_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$ and therefore in $G$.

To see that these groups are quasi-isometric, take the generating set for $G$ to be

$$S_G = \{(0,1)\} \cup \{(f_1, 0), (f_2, 0), (f_3, 0)\}$$

where $f_i : \mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z}$ is defined by

$$f_i(n) = \begin{cases} i & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Take the generating set for $H$ to be

$$S_H = \{(0,1)\} \cup \{(f_{(0,1)}, 0), (f_{(1,0)}, 0), (f_{(1,1)}, 0)\}$$

where $f_{(i,j)} : \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is given by

$$f_{(i,j)}(n) = \begin{cases} (i,j) & n = 0 \\ (0,0) & \text{otherwise.} \end{cases}$$

**Exercise.** Find a quasi-isometry $\mathrm{Cay}(G, S_G) \longrightarrow \mathrm{Cay}(H, S_H)$.

**Definition.** A group is *virtually* "blah" if it has a finite index subgroup which is "blah."

**Examples.** $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$ is virtually $\mathbb{Z}$.
$F_n \rtimes S_n$ is virtually free.

**Theorem 3.9.** *Let $G$ be a finitely generated group such that $\mathrm{Cay}(G) \simeq_{QI} \mathbb{Z}$. Then $G$ is virtually $\mathbb{Z}$.*

*Proof (Sketch).* First we check that there is an element of infinite order. We'll find $g \in G$ and $A \subset G$ such that $gA \subsetneq A$, and then $g^n \neq e$ for all $n \neq 0$.

Let $\varphi : \mathrm{Cay}(G) \longrightarrow \mathbb{R}$ be a quasi-isometry (since we assume $\mathrm{Cay}(G) \simeq_{\mathrm{QI}} \mathbb{Z}$ and $\mathbb{Z} \simeq_{\mathrm{QI}} \mathbb{R}$. $G$ acts on $\mathrm{Cay}(G)$ by isometries so any $g \in G$ determines a quasi-isometry $\psi_g : \mathbb{R} \longrightarrow \mathbb{R}$ making the following diagram commute:

$$\begin{tikzcd} \mathrm{Cay}(G) \arrow[r, "g"] \arrow[d, "\varphi"] & \mathrm{Cay}(G) \arrow[d, "\varphi"] \\ \mathbb{R} \arrow[r, "\psi_g"] & \mathbb{R} \end{tikzcd}$$

Now, either:

1. $\psi_g([0,\infty))$ is a bounded distance from $[\psi_g(0), \infty)$ or
2. $\psi_g([0,\infty))$ is a bounded distance from $(-\infty, \psi_g(0)]$.

(Recall that $X$ is at a *bounded distance from* $Y$ if there is an $m$ such that for all $x \in X$ there is a $y \in Y$ with $d(x,y) \leq m$, and for all $y \in Y$ there is an $x \in X$ with $d(x,y) < m$).

If (1) holds, then setting $A = V(\mathrm{Cay}(G)) \cap \varphi^{-1}([0,\infty))$, provided that $\varphi(g)$ is large enough then $gA \subsetneq A$. So, we need a $g$ such that $\psi_g([0,\infty))$ is a bounded distance from $[\psi_g(0), \infty)$ and $\psi_g(0) \gg 0$. To find such a $g$, take $h, k \in G$ such that $1, h, k$ are all far apart in $\mathrm{Cay}(G)$, so $\varphi(h), \varphi(1)$ and $\varphi(k)$ are all far apart in $\mathbb{R}$. Considering the images of $[0,\infty)$ under $\psi_1$, $\psi_h$,$\psi_k$, at least two of these images will be at bounded distance from each other, as there are only two ways of going off to $\infty$. So, at least two of $A$, $hA$ and $kA$ are nested. So, take $g$ to be one of the elements $h, k, hk^{-1}$ or $h^{-1}k$ depending on which way they overlap.

Let $H = \langle g \rangle$, and we will show that $[G : H] < \infty$. We know that $d(1, g^n) \longrightarrow \infty$ as $n \longrightarrow \pm\infty$ and $d(g^n, g^m) = d(1, g^{m-n})$. Take the quasi-isometry $\varphi : \mathrm{Cay}(G) \longrightarrow \mathbb{R}$ and define $f : \mathbb{Z} \longrightarrow \mathbb{R}$ by $n \mapsto \varphi(g^n)$. We then have:

1. For all $n$, $|f(n) - f(n+1)|$ is bounded;
2. For all $r \geq 0$, there is a $k \in \mathbb{N}$ such that $|f(n) - f(m)| \leq r$ implies that $|m - n| \leq k$.

**Exercise.** The image of such a map has the property that there is a $c$ such that for all $x \in \mathbb{R}$, there is an $n \in \mathbb{Z}$ such that $|x - f(x)| \leq c$.

Pulling things back yields that there is a $c'$ such that for all $g'$ in $G$, there is a $g'' \in H$ with $d(g', g'') \leq c'$. Therefore, $H$ has finite index in $G$. □

**Examples.**

- If $G, H$ are virtually abelian with $G \simeq_{\mathrm{QI}} H$ then $G, H$ are commensurable.
- Surprisingly, in the above only one of $G, H$ need be virtually abelian for the conclusion to hold.
- If $G, H$ are virtually free, and quasi-isometric then they are commensurable. Again, this actually only needs to be assumed for one of the groups.

# 4 Geometric Properties of Groups

**Definition.** Let $X \subseteq \mathbb{R}$ and let two functions $f, g : X \longrightarrow \mathbb{R}$. Write $f \prec g$ if there are constants $a, b > 0$ such that for some fixed $x_0$, for all $x \geq x_0$

$$f(x) \leq a g(bx).$$

If $f \prec g$ and $g \prec f$ write $f \asymp g$, and we call $f, g$ *equivalent*.

**Definition.** Let $X$ be a discrete metric space with a basepint $x_0$. The *growth function* is

$$\beta_{X,x_0}(r) = |\overline{B}(x_0, r)|.$$

**Lemma 4.1.** *The equivalence class under $\asymp$ of the growth function is invariant under quasi-isometry for groups.*

*Proof.* Exercise. □

In particular, $\beta_{G,g} = \beta_{G,h}$ for all $g, h \in G$, so we can drop the basepoint and just write $\beta_G$ for the growth function. We sometimes write $\beta_{G,S}$ if we wish to emphasise the generating set.

**Examples.**

1. $\beta_{\mathbb{Z}^k}(r) \asymp r^k$.

2. $\beta_{F(r)} \asymp (2k)^r$.

**Proposition 4.2.** *1. If $G$ is infinite, $\beta_{G,S}|_{\mathbb{N}}$ is increasing.*

*2. $\beta_{G,S}(r+t) \prec \beta_{G,S}(r)\beta_{G,S}(t)$.*

*3. $\beta_{G,S}(r) \prec |S|^r$.*

*Proof.* Exercise. ☐

Note that 1 implies that $\lim_{n\longrightarrow\infty} \beta_{G,S}(n)^{\frac{1}{n}}$ (which exists by Fekete's Lemma) is at least 1.

**Definition.** We say $G$ has *exponential growth* if $\lim_{n\longrightarrow\infty} \beta_{G,S}(n)^{\frac{1}{n}} > 1$. Otherwise, we say $G$ has *subexponential growth.*

**Examples.** A free group on at least 2 generators has exponential growth, whilst a finitely generated abelian group has subexponential growth.

**Proposition 4.3.** *Let $G$ be a finitely generated group. Then*

*1. If $H$ is a finitely generated subgroup of $G$, then $\beta_H \prec \beta_G$.*

*2. If $H$ is of finite index in $G$, $\beta_H \asymp \beta_G$.*

*3. If $N \lhd G$, $\beta_{G/N} \prec \beta_G$.*

*4. If $N \lhd G$ is a finite subgroup then $\beta_{G/N} \asymp \beta_G$.*

*Proof.* For 1, take $H = \langle T\rangle$ for $T$ finite and let $S$ be a finite generating set for $G$ containing $T$. Then $\mathrm{Cay}(H,T)$ is a subgraph of $\mathrm{Cay}(G,S)$ with

$$d_S(1,h) \leq d_T(1,h)$$

for all $h \in H$. So, $\overline{B}_H(1,r) \subset \overline{B}_G(1,r)$.

For 2,4 use Lemma 4.1 and Švarc-Milner.

For 3, let $G = \langle S\rangle$ for $S$ finite and set $\pi(S) = \{sN : s \in S, s \notin N\}$. Then $\langle\pi(S)\rangle = G/N$ and $\pi : G \longrightarrow G/N$ maps closed balls of radius $r$ about 1 onto closed balls of radius $r$ about 1 in $G/N$. ☐

There are two questions which arise early in the study of growth:

- Which types of growth are possible?
- Which groups have polynomial growth?

We know that virtually abelian groups have polynomial growth. In fact, we will show that nilpotent groups do.

Recall that a group is nilpotent if the lower central series

$$G > [G,G] > [[G,G],G] > \dots$$

eventually terminates with the trivial group.

**Proposition 4.4.** *Let $G$ be a 2-step nilpotent group, i.e. $[[G,G],G] = \{1\}$. Then $G$ has polynomial growth.*

*Proof.* Suppose $G$ is generated by $\{g_1, \dots, g_m\}$. We know that $[G,G]$ is abelian and lies in the centre $Z(G)$. Take a product of $n$ generators. Exchanging two generators produces a commutator on the right:

$$gh = hg[g^{-1}h^{-1}].$$

Commutators lie in the centre, so can move them all the way to the right of the product "at no cost." So we can arrange the generators in the following order: first all powers of $g_1$, then all powers of $g_2$, and so on, in at most $n^2$ moves. So, in $n^2$ moves we have produced a group element of the form

$$g_1^{\alpha_1} \dots g_n^{\alpha_n} C,$$

where $C$ is a product of at most $n^2$ commutators.

**Exercise.** $[G,G]$ is generated (in this case!) by $[g_i^{\pm 1}, g_j^{\pm 1}]$.

These commutators are words of length 1 in generators of $[G,G]$ which has polynomial growth of degree $D$, say. So $G$ has polynomial growth of degree bounded by $m + 2D$. □

**Theorem 4.5.** *All finitely generated virtually nilpotent groups have polynomial growth.*

To prove this, proceed by induction on the number of terms in the lower central series.

We might hope to generalize further.

**Definition.** A group $G$ is *solvable* is the derived series

$$G > [G,G] > [[G,G],[G,G]] > \dots$$

terminates with the trivial group after a finite number of steps.

**Example.** The group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ is solvable but not nilpotent. Let $a$ generate $\mathbb{Z}/2\mathbb{Z}$ and let $t$ generate $\mathbb{Z}$. Let $S = \{(0,t)\} \cup \{(f,0) \mid f(t^0) = a, f(t^i) = 1 \text{ if } i \neq 0\}$. Think about elements of the form $(0,t)\cdot(f,0)^{\alpha_1}\cdot(0,t)\cdot(f,0)^{\alpha_2}\dots(f,0)^{\alpha_n}\cdot(0,t)$.

Solvable groups do not have polynomial growth in general, and in fact

**Theorem 4.6** (Gromov 1981)**.** *A finitely generated group is of polynomial growth if and only if it is virtually nilpotent.*

**Remarks.** The proof uses asymptotic cones $(G, \frac{ds}{n})_n)$, Lie groups and Tits' alternative. For certain classes of group, a Tits-type alternative says that a group either has exponential growth or is "nice."

**Corollary 4.7.** *Being virtually nilpotent is a geometric property, that is, it is invariant under quasi-isometry.*

**Theorem 4.8** (Grigorchuk, 1983)**.** *There is a finitely generated group $G$ such that*

$$2^{r^{\alpha_1}} \leq \beta_G(r) \leq 2^{r^{\alpha_2}}$$

*where $0 < \alpha_1 < \alpha_2 < 1$, having intermediate growth. See de la Harpe.*

## 4.1 Ends

**Definition.** Let $X, Y$ be topological spaces. Recall that a map $f : X \longrightarrow Y$ is *proper* if $f^{-1}(C) \subseteq X$ is conpact whenever $C \subseteq Y$ is compact.

**Definition.** Let $X$ be a topological space. A *ray* in $X$ is a map $r : [), \infty) \longrightarrow X$. Let $r_1, r_2$ be proper rays. We say $r_1, r_2$ *converge to the same end* if for all compact $C \subset X$ and $N \in \mathbb{N}$ such that $r_1([N, \infty))$ and $r_2([N, \infty))$ are in the same connected component of $X \backslash C$. This defines an equivalence relation on rays and we call the equivalence class of a ray the *end*, written $\mathrm{end}(r)$. The set of equivalence classes is the *set of ends*, written $\mathrm{Ends}(X)$. If $|\mathrm{Ends}(X)| = m$, we say that $X$ has $m$ ends.

We can put a topology on $\mathrm{Ends}(X)$ by declaring $B$ to be closed if, for all $(r_n)$ with $\mathrm{end}(r_n) \in B$ for all $n$, with $\mathrm{end}(r_n) \longrightarrow \mathrm{end}(r)$ we have $\mathrm{end}(r) \in B$. We define convergence of ends as follows: $\mathrm{end}(r_n) \longrightarrow \mathrm{end}(r)$ if for any compact $C \subset X$ there is a sequence $N_n$ of integers such that $r_n([N_n, \infty))$ and $r([N_n, \infty))$ lie in the same path component of $X \backslash C$ for $n$ sufficiently large.

In what follows, a $k$-path from $x$ to $y$ in a metric space $X$ will be a sequence of points in $X$ with $x = x_1, x_2, \ldots, x_n = y$ such that $d(x_i, x_{i+1}) \leq k$ for all $i$.

**Lemma 4.9.** *Let $X$ be a proper geodesic metric space. Let $k > 0$ and let $r_1, r_2$ be proper rays in $X$. Let $\mathcal{G}_{x_0}(X)$ be the set of proper geodesic rays in $X$ issuing from $x_0$. Then*

1. $\mathrm{end}(r_1) = \mathrm{end}(r_2)$ *if and only if for all $R > 0$ there is a $T > 0$ such that $r_1(t)$ can be connected to $r_2(t)$ by a $k$-path in $X \backslash B(x_0, R)$ for all $t > T$.*

2. *The natural map $\mathcal{G}_{x_0,X} \longrightarrow \mathrm{Ends}(X)$ is surjective.*

*Proof.* 1. Every compact subset of $X$ is contained in some open ball about $x_0$ and vice versa, so can replace "compact subset" with "open ball" in the definition of ends, and can then concatenate geodesics from $x_i$ to $x_{i+1}$ to get a continuous path from a $k$-path.

2. Let $r : [0, \infty) \longrightarrow X$ be a proper ray. Let $c_n : [0, d_n] \longrightarrow X$ be a geodesic from $x_0$ to $r(n)$ and extend $c_n$ to the constant path $r(n)$ on $[d_n, \infty)$.

   Recall: The Arzelá-Ascoli Theorem says that if $Z$ is a compact metric space and $Y$ is a separable metric space then every sequence of equicontinuous functions $Y \longrightarrow Z$ has a subsequence that converges on compact subsets to a continuous function $f : Y \longrightarrow Z$.

Applying this to $(c_n)$: equicontinuity folllows from 1-Lipschitzness so there is a convergent subsequence of $c_n$ converging to a ray $c : [0, \infty) \longrightarrow X$, a geodesic ray with $\mathrm{end}(c) = \mathrm{end}(r)$.

□

**Exercise.** Let $X$ be a metric space. Given two maps $g, f : X \longrightarrow X$ we say they are *equivalent* and write $f \sim g$ if $\|f - g\|_\infty$ is finite. Show that the set of equivalence classes of quasi-isometries of $X$ form a group.

We call this group the *quasi-isometry group* of $X$ and write it $\mathrm{QI}(X)$.

**Exercise.** Show that a quasi-isometry$\varphi : X \longrightarrow Y$ induces an isomorphism $\varphi_* : \mathrm{QI}(X) \longrightarrow \mathrm{QI}(Y)$.

**Proposition 4.10.** *Let $X, Y$ be proper geodesic metric spaces. Every quasi-isometry $f : X \longrightarrow Y$ induces a homeomorphism $\overline{f} : \mathrm{Ends}(X) \longrightarrow \mathrm{Ends}(Y)$. Moreover, the map*

$$\begin{aligned} \mathrm{QI}(X) &\longrightarrow \mathrm{Homeo}(\mathrm{Ends}(X)) \\ f &\longmapsto \overline{f} \end{aligned}$$

*is a homeomorphism.*

*Proof.* Let $r$ be a geodesic ray in $X$ and let $f * r$ be a ray in $Y$ obtained by concatenating some choice of geodesic segments, $[f(r(n)), f(r(n+1))]$ for $n \in \mathbb{N}$. If $f$ is a quasi-isometry then $f * r$ is a proper ray, and $\mathrm{end}(f * r)$ is independent of the choice of geodesic segments.

Set

$$\begin{aligned} \overline{f} : \mathrm{Ends}(X) &\longrightarrow \mathrm{Ends}(Y) \\ \mathrm{end}(r) &\longmapsto \mathrm{end}(f * r) \end{aligned}$$

for every geodesic ray in $\mathrm{Ends}(X)$. Part 2 of Lemma 4.8 now gives that $\overline{f}$ is defined on all of $\mathrm{Ends}(X)$.The imge of a $k$-path under $f$ is a $(\lambda k + c)$-path (if $f$ is a $(\lambda, c)$-quasi-isometry), so $\overline{f}$ is well defined by Lemma 4.8 part 1.

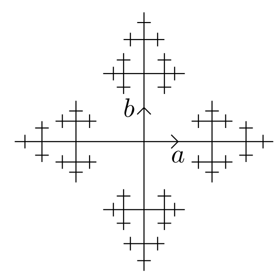
**Exercise.** Check that $\overline{f}$ is continuous. Check further that if $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ are quasi-isometries then $\overline{g}\,\overline{f} = \overline{(gf)}$.

**Exercise.** Check that if $g$ is the quasi-inverse of $f$ (i.e. $gf$ is trivial in $\mathrm{QI}(X)$) then $\overline{g}\,\overline{f}$ is the identity map in $\mathrm{Ends}(X)$.

□

**Definition.** Let $G$ be a finitely generated group. Then $\mathrm{Ends}(G)$ is defined to be $\mathrm{Ends}(\mathrm{Cay}(G))$.

**Examples.** $F_n$ for $n \geq 2$ has infinitely many ends.




Any group which is virtually $\mathbb{Z}$ has two ends.
Lots of groups have one end, for example $\mathbb{Z}^2$.
Finite groups do not have any ends.

**Theorem 4.11.** *Let $G$ be a finitely generated group.*

1. *$G$ has zero, one, two or infinitely many ends.*

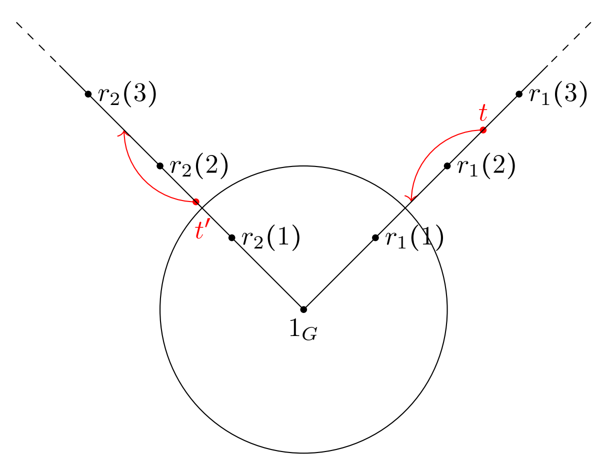
2. *$G$ has no ends if and only if it is finite.*

3. *$G$ has two ends if and only if it is virtually $\mathbb{Z}$.*

4. *$G$ has infinitely many ends if and only if it can be expressed as an amalgamated free product over a finite group $C$ as $A *_C B$ or as an HNN extension $A*_C$ with $|A/C| \geq 3$ and $|B/C| \geq 2$.*

1–3 are due to Hopf in 1943, and 4 is due to Stallings in 1968. We just prove 1.

*Proof.* Let $G$ act on $\mathrm{Cay}(G,S)$ by isometries. This gives a homomorphism $G \longrightarrow \mathrm{Homeo}(\mathrm{Ends}(\mathrm{Cay}(G)))$ by Proposition 4.9. Let $H$ be the kernel. If $\mathrm{Ends}(\mathrm{Cay}(G))$ is finite then $H$ is of finite index in $G$. Assume for contradiction that $\infty > [G : H] \geq 3$. Let $e_0, e_1, e_2 \in \mathrm{Ends}(X)$ be distinct. Fix two geodesic rays $r_1, r_2 : [0,\infty) \longrightarrow \mathrm{Cay}(G)$ with $r_1(0) = r_2(0) = 1_G$ and $\mathrm{end}(r_i) = e_i$.



Since $H$ is of finite index in $G$ there is a $\mu$ such that for all $g \in G$ there is an $h \in H$ with $d(g,h) \leq \mu$. So there is a proper ray $r_0 : [0,\infty) \longrightarrow \mathrm{Cay}(G)$ with $\mathrm{end}(r_0) = e_0$. Then, $d(r_0(n), 1_G) \geq n$ and $r_0(n) \in H$ for all $n$. Set $h_n = r_0(n)$.

Fix $N > 0$ such that $r_0[N,\infty), r_1[N,\infty), r_2[N,\infty)$ all lie in different path components of $\mathrm{Cay}(G)\backslash B(1_G, N)$. If $t, t' > 2N$ then

$$d(r_1(t), r_2(t')) > 2N$$

since any path joining $r_1(t), r_2(t')$ must pass through $B(1_G, N)$. $H$ acts trivially on $\mathrm{Ends}(\mathrm{Cay}(G))$ and so $\mathrm{end}(h_n r_i) = \mathrm{end}(r_i)$ for $i = 1, 2$. Let $n > 3N$, then $h_n r_i(0) = h_n$ lies in a different path component to $r_i[N,\infty)$ so $h_n(r_i)$ must pass through $B_G(1_G, N)$. Therefore there is some point on that say, $t$, such that $h_n r_1(t) \in B(1_G, N)$ and similarly there is a $t'$ such that $h_n)r_2(t')) \in B(1_G, N)$.

Since $h_n$ acts by isometries, this gives us that

$$\begin{aligned} 2N &> d(h_n(r_1(t), h_n(r_2(t'))) \\ &= d(r_1(t), r_2(t')) \\ &> 2N, \end{aligned}$$

which is a contradiction. □

Other geometric properties include:

- being virtually free (which is equivalent to being quasi-isometric to a tree)
- being finitely presentable (see Bridson–Haefliger Proposition 8.24.

# 5 Amenability

**Definition.** Let $G$ be a group acting on a set $X$ and let $A, B \subseteq X$. We say that $A, B$ are (finitely) *$G$-equidecomposable* if there are partitions

$$\begin{aligned} A &= A_1 \sqcup A_2 \sqcup \ldots \sqcup A_n \\ B &= B_1 \sqcup B_2 \sqcup \ldots \sqcup B_n \end{aligned}$$

such that there are $g_1, \ldots, g_n \in G$ with $g_i(A_i) = B_i$ for all $i$. If such partitions exist, write $A \sim B$. A *realization* $h$ of $A \sim B$ is a bijection $h : A \to B$ such that a decomposition as above exists with $h(a_i) = g_i a_i$.

**Notation.** If $A \sim C$ for $C \subseteq B$, write $A \lesssim B$.

For fixed $G$, equidecomposability is an equivalence relation. To see the transitivity, let $A \sim B$ and $B \sim D$. Let the relevant partitions be

$$A = \bigsqcup_{i=1}^{n} A_i, \; B = \bigsqcup_{i=1}^{n} B_i = \bigsqcup_{i=1}^{m} C_i \text{ and } D = \bigsqcup_{i=1}^{m} D_i$$

We can form partitions $A_{ij} = g_i^{-1}(B_i \cap C_j)$ and $D_{ij} = h_j g_i(A_{ij})$. Then $A \sim D$ via $\{h_j g_i\}$.

**Theorem 5.1.** *Let $G$ act on $X$ and let $A, B \subseteq X$. If $A \precsim B$ and $B \precsim A$ then $A \sim B$.*

*Proof.* Take $f : A \longrightarrow B_1$ and $g : A_1 \longrightarrow B$ with $A_1 \subseteq A$, $B_1 \subseteq B$. Define $C_0 = A \backslash A_1$ and iteratively set $C_{n+1} = g^{-1} f(C_n)$ for each $n \geq 0$.

Set $C = \cup_{i=1}^{\infty} C_i$. Then, if $a \in A \backslash C$, $a \notin C_n$ for all $n$ and therefore $g(a) \notin f(C_n)$. Consequently, $g(A \backslash C) \subseteq B \backslash f(C)$. Similarly, $B \backslash f(C) \subseteq g(A \backslash C)$. So, $A \backslash C \sim B \backslash f(C)$. Clearly, $C \sim f(C)$ so sticking these together, $A \sim B$. ☐

**Corollary 5.2.** *The following are equivalent.*

1. *There are proper disjoint subsets $A, B \subset X$ such that $A \sim X \sim B$.*
2. *There are proper disjoint subsets $A, B \subset X$ such that $A \sim X \sim B$ and $X = A \sqcup B$.*

*Proof.* To see that 1 implies 2, since $X \sim B \subseteq X \backslash A \subset X$ we have $X \precsim (X \backslash A)$, and we know $(X \backslash A) \subset X$. So, $X \backslash A \precsim X$ so by Theorem 5.1, $A \sim X \sim X \backslash A$. ☐

**Definition.** Let $G$ be a group acting on $X$. If the statements in corollary 5.2 hold, then we say $X$ is (finitely) $G$-paradoxical.

**Proposition 5.3.** 1. *With the left multiplication action, $F_2$ is $F_2$-paradoxical.*

2. *If $F_2$ acts freely on a set $S$, then $X$ is $F_2$-paradoxical.*

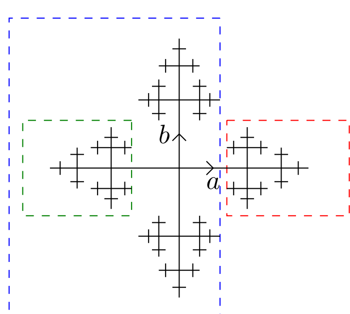
*Proof.* 1. Let $F_2 = \langle a, b \rangle$. For $y \in \{a, a^{-1}, b, b^{-1}\}$, let

$$W(y) = \{\text{set of reduced words in } F_2 \text{ beginning with } y\}.$$

Then

$$\begin{aligned} F_2 &= \{1\} \sqcup W(a) \sqcup W(b) \sqcup W(b^{-1}) \\ &= W(a) \sqcup aW(a^{-1}) \\ &= W(b) \sqcup bW(b^{-1}) \end{aligned}$$

On the Cayley graph, we get the following picture, with $aW(a^{-1})$ contained in the blue rectangle, $W(a)$ in the red and $W(a^{-1})$ in the green.



We see that, for

$$A = W(a) \sqcup W(a^{-1}),\ B = W(b) \sqcup W(b^{-1}),$$

we have $A \sim F_2 \sim B$ which satisfies the first statement in Corollary 5.2.

2. Take $M$ to be a set of representatives of $F_2$ orbits of $X$. For $c \in F_2$, set

$$X_c = \{z_m \mid z \in W(c), m \in M\}.$$

Then $X_a$, $X_b$ $X_{a^{-1}}$ and $X_{b^{-1}}$ are all disjoint and we have that

$$\begin{aligned} X &= X_a \sqcup aX_{a^{-1}} \\ &= X_b \sqcup bX_{b^{-1}} \end{aligned}$$

which gives a paradoxical decomposition of $X$ as before.

□

**Proposition 5.4.** *The special orthogonal group* $\mathrm{SO}(3,\mathbb{R})$ *contains* $F_2$ *as a subgroup.*

*Proof.* The matrices

$$\sigma = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{-2\sqrt{2}}{3} & \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix} \text{ and } \rho = \begin{bmatrix} \frac{1}{3} & \frac{-2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

generate $F_2$. □

**Theorem 5.5.** *There is a* $D \subset S^2$ *such that* $S^2 \backslash D$ *is* $\mathrm{SO}(3,\mathbb{R})$*-paradoxical.*

*Proof.* Every nontrivial element in $\mathrm{SO}(3,\mathbb{R})$ fixes exactly two points in $S^2$. Let $D$ be the union of the fixed points under the action of $F_2 < \mathrm{SO}(3,\mathbb{R})$. Then $F_2$ acts freely on $S^2 \backslash D$ and applying Proposition 5.3, $S^2 \backslash D$ is $F_2-$*paradoxical* (and hence $\mathrm{SO}(3,\mathbb{R})$-paradoxical). □

**Proposition 5.6.** *For a countable* $D \subset S^2$*, the sets* $S^2$ *and* $S^2 \backslash D$ *are* $\mathrm{SO}(3,\mathbb{R})$*-equidecomposable.*

*Proof.* Let $l$ be a line through the origin missing $D$ (which exists since $D$ is countable). Using countability again, there is a $\theta$ such that for all $n \in \mathbb{Z}_{\geq 0}$ the image of $\rho^n(D)$ of $D$ under rotation by $n\theta$ about $l$ does not intersect $D$.

Set

$$\overline{D} = \bigcup_{n=0}^{\infty} \rho^n(D).$$

Then

$$\begin{aligned} S^2 &= \overline{D} \sqcup S^2 \backslash \overline{D} \\ &\sim \rho(\overline{D}) \sqcup S^2 \backslash D \\ &= S^2 \backslash D. \end{aligned}$$

□

Thus by 5.5, 5.6 and transitivity of ~ we get

**Theorem 5.7** (Banach-Tarski)**.** *$S^2$ is $\mathrm{SO}(3,\mathbb{R})$-paradoxical.*

**Exercise.** $S^n$ is $\mathrm{SO}(n+1,\mathbb{R})$-paradoxical for all $n \geq 2$.

One consequence of this is that we cannot put finitely additive probability measures that are invariant under rotations on subsets of $S^n$.

**Theorem 5.8** (Banach-Tarski Paradox)**.** *Let $E(3)$ be the group of isometries of $\mathbb{R}^3$. Any solid ball in $\mathbb{R}^3$ is $E(3)$-paradoxical. $\mathbb{R}^3$ is $E(3)$-paradoxical.*

*Proof.* Exercise. □

**Theorem 5.9** (Tarski)**.** *Let $G$ act on $X$ and let $E \subseteq X$. There is a finitely additive measure $\mu : P(X) \longrightarrow [0,\infty)$ with $\mu(E) = 1$ that is $G$-invariant if and only if $E$ is not $G$-paradoxical.*

## 5.1 Non-paradoxical groups: amenability

**Definition.** Let $G$ be a discrete (respectively locally compact) group. A *measure* on $G$ is a finitely additive measure $\mu$ on the power set of $G$ (respectively on Borel sets of $G$) with $\mu(G) = 1$ and which is left-invariant, that is $\mu(gA) = \mu(A)$ for all $g \in G$, $A \subseteq G$.

$G$ is *amenable* if it admits such a measure.

Clearly if the action of $G$ on itself by left multiplication is paradoxical, $G$ is not amenable. So, $F_2$ (or any group containing $F_2$ as a subgroup) is not amenable.

The Von-Neumman conjecture stated that all non-amenable groups contain $F_2$ as a subgroup. This was disproved by Olshanski, who gave the example of the "Tarski monsters." For $p$ a fixed prime, these have the property that every nontrivial subgroup of $G$ either has order $p$ or is $G$ itself.

**Definition.** Let $G$ be a finitely generated group, and work in $l^\infty(G)$, Let $M$ be a linear functional

$$M : l^\infty(G) \longrightarrow \mathbb{C}.$$

If

1. $M(f) \geq 0$ if $f(g) \geq 0$ for all $g \in G$,
2. $M(\chi_G) = 1$ if $\chi_G \equiv 1$ on $G$ and
3. $M(g(f)) = M(f)$ for all $g \in G$ where $g(f)(h) = f(g^{-1}h)$

then $M$ is called a *left-invariant mean* on $G$.

**Proposition 5.10.** *A group $G$ is amenable if and only if it admits a left-invariant mean.*

*Proof.* If $G$ is amenable, define

$$M(f) = \int f d\mu.$$

Conversely, if $G$ has a left-invariant mean $M$ define

$$\mu(A) = M(\chi_A).$$

□

**Proposition 5.11.** *Let $G$ act on $X$ and let $G$ be amenable. Then there is a finitely additive probability measure on $X$ that is $G$-invariant. In particular (using 5.9), $X$ is not $G$-paradoxical.*

*Proof.* Let $\mu$ be the measure of $G$ realising amenability on $G$. Fix $x_0 \in X$ and define

$$\begin{aligned} \nu : P(X) &\longrightarrow [0,1] \\ A &\longmapsto \mu\{g \in G \mid gx_o \in A\} \end{aligned}$$

□

**Theorem 5.12.** *The following are equivalent.*

1. *$G$ is amenable.*
2. *$G$ admits a left-invariant mean.*
3. *$G$ is not paradoxical.*

**Example.** All finite groups are amenable, by taking $\mu(A) = \frac{|A|}{|G|}$.

**Proposition 5.13.** 1. *If $G$ is amenable and $H \leq G$ then $H$ is amenable.*

2. *If $G$ is amenable and $N \lhd G$ then $G/N$ is amenable.*
3. *If $N$ and $G/N$ are amenable then $G$ is amenable.*
4. *If $\{G_i\}$ is a directed system of amenable groups indexed by $I$ then $\bigcap_{i \in I} G_i$ is amenable.*

*Proof.* Let $\mu : P(G) \longrightarrow [0,1]$ realise the amenability of $G$.

1. Any $H \leq G$ has a right transversal $M$ in $G$. Define

$$\begin{aligned} \nu : P(G) &\longrightarrow [0,1] \\ A &\longmapsto \mu(AM). \end{aligned}$$

   This gives the desired measure on $H$.

2. Define

$$\begin{aligned}\lambda : P(G/N) &\longrightarrow [0,1]\\ A &\longmapsto \mu(AN).\end{aligned}$$

   This gives the desired measure on $G/N$.

3. Exercise.

4. For each $i, j \in I$ there is a $k$ such that $G_i, G_j$ are contained in $G_k$. For each $i$ write $\mu_i$ for the measure realising amenability of $G_i$. Define

$$M_i = \{\mu : P(G) \longrightarrow [0,1] \mid \ \mu \text{ finitely additive, } \mu(gA) = \mu(A) \ \forall g \in G_i\}.$$

Then $M_i$ is nonempty since the measure given by $\mu(A) = \mu_i(A \cap G_i)$ is in $M_i$. Moreover, by Tychonoff's Theorem, $[0,1]^{P(G)}$ is compact with respect to the product topology, and $M_i$ is closed in $[0,1]^{P(G)}$. If $G_i, G_j \subseteq G_k$ then $M_k \subseteq M_i \cap M_j$.

So, the family of closed subsets $\{M_i\}_{i \in I}$ has the finite intersection property (any finite intersection of the $M_i$ is nonempty. But the finite intersection property for closed sets in a compact space implies that $\bigcap_{i \in I} M_i$ is nonempty, and this intersection contains a measure realising the amenability of $G$. □

## 5.2 Geometric Viewpoint of Amenability

**Definition.** A finitely generated group satisfies the *Følner condition* if for every finite subset $A \subseteq G$, for all $\epsilon > 0$ there is a nonempty finite $F \subseteq G$ such that

$$\frac{|aF \Delta F|}{|F|} \leq \epsilon$$

for all $a \in A$.

**Theorem 5.14.** *For a finitely generated group $G$, the following are equivalent.*

1. *$G$ is amenable.*
2. *$G$ satisfies the Følner condition.*

*Proof.* Suppose the second statement holds. For $A \subseteq G$ and $\epsilon > 0$ define the set $M_{A,\epsilon}$ to be the set of finitely additive probability measures $\mu$ on $G$ such that $|\mu(B) - \mu(aB)| < \epsilon$ for all $B \subseteq G$, $a \in A$.

Then $M_{A,\epsilon}$ is closed in $[0,1]^{P(G)}$ and we jut need to check that it is nonempty. Define

$$\mu(B) = \frac{|B \cap F|}{|F|},$$

where $F$ is given by the Følner condition for $A, \epsilon$. Then we have

$$
\begin{aligned}
|\mu(B) - \mu(aB)| &= |\frac{|B \cap F}{|F|} - \frac{|aB \cap F|}{|F|}| \\
&\leq \frac{|(B \Delta aB) \cap F|}{|F|} \\
&\leq \frac{|F \Delta aF|}{|F|} \\
&\leq \epsilon.
\end{aligned}
$$

So, $M_{A,\epsilon}$ is nonempty and $\{M_{A,\epsilon}\}$ satisfies the finite intersection property, so there is a $\mu \in \bigcap_{A,\epsilon} M_{A,\epsilon}$ realising the amenability of $G$.

For the other direction see Theorem 16.62 of Druţu-Kapovich or Theorem 4.2.3 of Juschenko, "Amenability". □

## 5.3 Geometric Interpretation of the Følner condition

**Definition.** Let $X$ be a graph. The *Cheeger constant* $h(X)$ is defined by

$$h(X) = \inf \frac{|\partial A|}{|A|}$$

where the infimum is taken over all finite nonempty subsets $A$ of $V(X)$, and the boundary of $A$ is $\partial A = \{x \in X \backslash A \mid x \text{ is connected to some } a \in A \text{ by an edge}\}$.

Note that:

- If $X$ is finite then $h(X) = 0$.
- If $X$ is infinite, it is possible that $h(X) = 0$.
- If $T$ is a tree then $h(T) \neq 0$.

**Proposition 5.15.** *Let $G$ be a finitely generated group. The following are equivalent.*

1. *$G$ satisfies the Følner condition (i.e. $G$ is amenable).*
2. *$h(\mathrm{Cay}(G, S)) = 0$ for all generating sets $S$.*
3. *$h(\mathrm{Cay}(G, S)) = 0$ for some generating set $S$.*

*Proof.* It is an exercise to check that 3 implies 1. 2 implies 3 is immediate so we just do 1 implies 2.

First, note that we can replace $\frac{|F\Delta aF|}{|F|}$ by $\frac{|F\Delta Fa|}{|F|}$ which is seen easily by using inverses. Given $A \subseteq G$ and $\epsilon > 0$ take the Følner condition's promised set $F$ corresponding to $A^{-1} = \{a^{-1} \mid a \in A\}$ and $\epsilon$.

Then, $F^{-1}$ satisfies

$$\frac{|F^{-1}a\Delta F^{-1}|}{|F^{-1}|} = \frac{|a^{-1}F\Delta F|}{|F|} \leq \epsilon.$$

Now take $A = S^{\pm 1}$, with $S$ a generating set for $G$ and fixed $\epsilon > 0$. Let $F \subseteq G$ be such that for all $s \in S^{\pm 1}$ we have

$$\frac{|Fs\Delta F|}{|F|} \leq \epsilon.$$

Then

$$\begin{aligned}
\frac{|\partial F|}{|F|} &= \frac{|\{gs \mid g \in F, s \in S^{\pm 1}, gs \notin F\}|}{|F|} \\
&\leq \frac{|\bigcup_{S^{\pm 1}} F_s \Delta F|}{|F|} \\
&\leq |S^{\pm 1}|\epsilon \\
&\leq 2|S|\epsilon.
\end{aligned}$$

Therefore, taking the infimum,

$$\inf \frac{|\partial F|}{|F|} = 0.$$

$\square$

**Corollary 5.16.** *All finitely generated groups which have subexponential growth are amenable.*

*Proof.* Exercise. $\square$

**Corollary 5.17.** *All solvable groups are amenable.*

*Proof.* By 5.16 all abelian groups are amenable, so apply 5.13. $\square$

**Corollary 5.18.** *Amenability is invariant under quasi-isometry.*

*Proof.* Exercise. $\square$

We finish with a question. The class of groups constructed using finite and abelian groups are all amenable (by 5.13). These are the *elementary amenable groups.* Is the set of elementary amenable groups closed under quasi-isometry?

# References

[BH99] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999. MR1744486

[DK18] Cornelia Druţu and Michael Kapovich, *Geometric group theory*, American Mathematical Society Colloquium Publications, vol. 63, American Mathematical Society, Providence, RI, 2018. With an appendix by Bogdan Nica. MR3753580

[dlH00] Pierre de la Harpe, *Topics in geometric group theory*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2000. MR1786869

[Sta68] John R. Stallings, *On torsion-free groups with infinitely many ends*, Ann. of Math. (2) **88** (1968), 312–334, DOI 10.2307/1970577. MR228573